Discrete conservation laws and port-Hamiltonian systems on graphs and complexes

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Abstract

In this paper we present a unifying geometric framework for modeling various sorts of physical network dynamics as port-Hamiltonian systems. Basic idea is to associate with the incidence matrix of the graph a Dirac structure relating the flow and effort variables associated to the edges, internal vertices, and boundary vertices of the graph. This Dirac structure captures the basic conservation/balance laws of the system. Examples from different origins such as consensus algorithms and coordination control strategies for multi-agent systems share the same structure. The framework is extended to k-complexes primarily motivated by the discretization of continuous conservation laws.

1 Introduction

Discrete topological structures arise abundantly in physical systems modeling. A classical approach to modeling and analysis of electrical circuits, dating back to Kirchhoff, is through the circuit graph. Similar approaches apply to many other cases, including e.g. mass-spring-damper mechanical systems, multi-body systems, hydraulic networks, chemical reaction networks, and power systems. Common feature of all these cases is that the

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discrete structures, in particular graphs, are blended with dynamical relations, leading to various sorts of *network dynamics*.

During the last two decades network dynamics has received ever-increasing attention, with inputs, among others, from graph theory, multi-agent systems, dynamical systems, as well as statistical mechanics. In this paper we wish to lay down a general *geometric framework for defining physical dynamics on directed graphs*. We will restrict attention to *fixed* directed graphs, that is, we do *not* study the possible dynamics of the graphs themselves, as e.g. studied in random graph theory or statistical mechanics. Furthermore, our graphs physically correspond to some sort of *conservation* (or *balance* laws), relating real-valued variables associated to the vertices to those corresponding to the edges of the graph. Dynamics arises because of *storage* (accumulation) of quantities corresponding to some of these variables, either at the vertices or at the edges (or both). Finally, in order to capture interaction with the environment and interconnection to other networks we explicitly consider *boundary* vertices, corresponding to the *terminals* of the network.

The generalized Hamiltonian nature of the resulting dynamical models is due to the common assumption that the flux variables corresponding to storage at the vertices and/or edges are derivable from some energy function (defining the energy corresponding to storage). Next to these *energy-storage* constitutive relations, the remaining internal variables will be related by static *energy-dissipating* constitutive relations. This will imply that the total stored energy (the Hamiltonian function) itself satisfies a conservation law: the increase of the total energy is equal to externally supplied power (through the boundary vertices of the graph), minus the power lost in the dissipative elements (corresponding to some of the edges or vertices of the graph). The resulting systems fall within the class of *port-Hamiltonian systems*, as coined and explored in e.g. [35, 32, 36, 13]. They are truly *generalized* Hamiltonian systems, since they allow for energy-dissipation and interaction with the environment.

From a geometric point of view the generalized Hamiltonian structure of the network dynamics is defined, apart from its Hamiltonian function and energy-dissipative relations, by a *Dirac structure*. This Dirac structure (generalizing the symplectic or Poisson structure from classical mechanics) is directly defined by the *incidence* matrix of the directed graph, and thus captures the conservation laws. In fact, we will show how a directed graph gives rise to three canonically defined Dirac structures on its vertex and edge spaces. The first two of them only differ in the different role of the boundary vertices, while the third, the Kirchhoff-Dirac structure, captures the special

case where no storage or dissipation is associated with the vertices of the graph (corresponding to Kirchhoff's current laws).

We will illustrate this framework on some of the physical examples mentioned above. Furthermore, we will show how the same port-Hamiltonian structure is shared by network dynamics with a different origin, such as consensus and coordination control algorithms.

While perhaps all examples given in the paper are relatively simple, and can be approached from other angles as well, we believe that a major contribution of the paper resides in the identification of a common mathematical structure in all these examples, which is moreover closely related to classical Hamiltonian and gradient dynamical systems.

In the last part of the paper we will sketch how the geometric framework as developed for directed graphs can be extended to arbitrary k-complexes. By considering standard simplicial complexes this allows a structure-preserving spatial discretization of distributed-parameter physical systems, otherwise described by partial differential equations. This is illustrated on the 2-D Maxwell equations and a general form of discretized diffusive systems. Furthermore, we explore its potential fro constructing generalized consensus algorithms.

Preliminary work regarding Sections 3.4 and 3.5 can be found in [39], and regarding Sections 5.1, 5.2 and 6.2 in [37, 38].

2 From directed graphs to Dirac structures

In this section we will define two, closely related, canonical Dirac structures on the combination of the vertex, edge and boundary spaces of a directed graph, and their dual spaces. Before doing so we first recall some basic notions of graph theory and Dirac structures.

2.1 Directed graphs and their vertex and edge spaces

First we recall some standard definitions regarding directed graphs, as can be found e.g. in [5]. A *directed graph* [1] $\mathcal{G}$ consists of a finite set $\mathcal{V}$ of *vertices* and a finite set $\mathcal{E}$ of directed *edges*, together with a mapping from $\mathcal{E}$ to the set of ordered pairs of $\mathcal{V}$, where no self-loops are allowed. Thus to any branch $e \in \mathcal{E}$ there corresponds an ordered pair $(v, w) \in \mathcal{V} \times \mathcal{V}$ (with $v \neq w$), representing the tail vertex v and the head vertex w of this edge.

[1] Sometimes called a *multi-graph* since we allow for the existence of multiple branches between the same pair of vertices.

A directed graph is completely specified by its *incidence matrix* $\hat{B}$, which is an $N \times M$ matrix, N being the number of vertices and M being the number of edges, with (i,j)-th element equal to -1 if the j-th edge is an edge towards vertex i, equal to 1 if the j-th edge is an edge originating from vertex i, and 0 otherwise. Since we will only consider directed graphs in this paper 'graph' will throughout mean 'directed graph' in the sequel.

Given a graph, we define its *vertex space* Λ_0 as the vector space of all functions from $\mathcal{V}$ to some linear space $\mathcal{R}$. In the examples, $\mathcal{R}$ will be mostly $\mathcal{R} = \mathbb{R}$ or $\mathcal{R} = \mathbb{R}^3$. In the first case, Λ_0 can be identified with $\mathbb{R}^N$. Furthermore, we define its *edge space* Λ_1 as the vector space of all functions from $\mathcal{E}$ to the same[2] linear space $\mathcal{R}$. Again, if $\mathcal{R} = \mathbb{R}$ then Λ_1 can be identified with $\mathbb{R}^M$.

The dual spaces of Λ_0 and Λ_1 will be denoted by Λ^0, respectively Λ^1.

Remark 2.1. *Hence Λ^0 can be represented as the linear space of functions from $\mathcal{V}$ to the dual linear space $\mathcal{R}^*$, and Λ^1 as the linear space of functions from $\mathcal{E}$ to $\mathcal{R}^*$. Thus the duality pairing between $f \in \Lambda_0$ and $e \in \Lambda^0$ is given as*

$$< f \mid e >= \sum_{v \in \mathcal{V}} < f(v) \mid e(v) >,$$

where $< \dot{\mid} \dot{>}$ on the right-hand side denotes the duality pairing between $\mathcal{R}$ and $\mathcal{R}^*$, and a similar expression holds for $f \in \Lambda_1$ and $e \in \Lambda^1$.

The incidence matrix $\hat{B}$ of the graph induces a linear map B from the edge space to the vertex space as follows. Define $B : \Lambda_1 \to \Lambda_0$ as the linear map with matrix representation

$$\hat{B} \otimes I$$

where $I : \mathcal{R} \to \mathcal{R}$ is the *identity map and* $\otimes$ denotes the Kronecker product. B will be called the *incidence operator*. For $\mathcal{R} = \mathbb{R}^3$ the incidence operator is given in matrix form as $\hat{B} \otimes I_3$, with I_3 the 3×3-identity matrix, while for $\mathcal{R} = \mathbb{R}$ the incidence operator reduces to the linear map given by the matrix $\hat{B}$ itself. *In this case ($\mathcal{R} = \mathbb{R}$) we will throughout use B both for the incidence matrix and for the incidence operator.*

The adjoint map of B is denoted as

$$B^* : \Lambda^0 \to \Lambda^1,$$

[2]In principle we could also associate with the edges a linear space $\mathcal{R}'$ which is *different* from the space $\mathcal{R}$ associated with the vertices. In that case the definition of the incidence operator needs an additional linear map from $\mathcal{R}'$ to $\mathcal{R}$.

and is called the *co-incidence* operator. For $\mathcal{R} = \mathbb{R}^3$ the co-incidence operator is given by $\hat{B}^T \otimes I_3$, while for $\mathcal{R} = \mathbb{R}$ the co-incidence operator is simply given by the transposed matrix $\hat{B}^T$. *In the latter case we will throughout use B^T both for the co-incidence matrix and the for co-incidence operator.*

We will use the terminology[3] *flows* for the elements of Λ_0 and Λ_1 (notation f_0 and f_1), and *efforts* for the elements of their dual spaces Λ^0 and Λ^1 (notation e^0, respectively e^1).

2.2 Open graphs

An *open graph* $\mathcal{G}$ is obtained from an ordinary graph with set of vertices $\mathcal{V}$ by identifying a subset $\mathcal{V}_b \subset \mathcal{V}$ of N_b *boundary vertices*. The interpretation of $\mathcal{V}_b$ is that these are the vertices that are open to interconnection (i.e., with other open graphs). The remaining subset $\mathcal{V}_i := \mathcal{V} - \mathcal{V}_b$ are the N_i *internal vertices* of the open graph.

The splitting of the vertices into internal and boundary vertices induces a splitting of the vertex space and its dual, given as

$$\begin{aligned} \Lambda_0 &= \Lambda_{0i} \oplus \Lambda_{0b} \\ \Lambda^0 &= \Lambda^{0i} \oplus \Lambda^{0b} \end{aligned}$$

where Λ_{0i} is the vertex space corresponding to the internal vertices and Λ_{0b} the vertex space corresponding to the boundary vertices. Consequently, the incidence operator $B : \Lambda_1 \to \Lambda_0$ splits as

$$B = B_i \oplus B_b$$

with $B_i : \Lambda_1 \to \Lambda_{0i}$ and $B_b : \Lambda_1 \to \Lambda_{0b}$.

Furthermore, we will define the *boundary space* Λ_b as the linear space of all functions from the set of boundary vertices $\mathcal{V}_b$ to the same [4] linear space $\mathcal{R}$. Note that the boundary space Λ_b is *isomorphic* to the linear space Λ_{0b}, and that using this isomorphism the linear mapping B_b can be also regarded as a mapping

$$B_b : \Lambda_1 \to \Lambda_b$$

[3] This terminology stems from port-based and bond-graph modeling [26], where it has a slightly more specific connotation then in our case. The space Λ_0 is also called the space of 0-chains, while the elements of Λ_1 are called the 1-chains. Furthermore, the dual spaces Λ^0 and Λ^1 are called the space of 0-cochains, respectively 1-cochains.

[4] Again, in principle we could also associate with the boundary vertices a linear space which is *different* from the space $\mathcal{R}$ associated with the vertices and the edges.

called the *boundary incidence operator*. Nevertheless, we will be careful in distinguishing the two isomorphic linear spaces Λ_b and Λ_{0b} because of their different interpretations in physical examples. The dual space of Λ_b will be denoted as Λ^b. The elements $f_b \in \Lambda_b$ are called the *boundary flows* and the elements $e^b \in \Lambda^b$ the *boundary efforts*.

Remark 2.2. *Another interesting case, which for clarity of exposition will not be considered in this paper, is to allow for interaction between graphs through boundary edges.*

2.3 Dirac structures

Recall ([35, 8, 32]) the definition of a (constant[5]) Dirac structure. Consider a vector space $\mathcal{F}$ with dual space $\mathcal{F}^*$. As before, the variables $f \in \mathcal{F}$ are called the *flow* variables, while the conjugate variables $e \in \mathcal{F}^*$ are called the *effort* variables. Define on the total space $\mathcal{F} \times \mathcal{F}^*$ the *indefinite inner product* $\ll \cdot, \cdot \gg$ as

$$\ll (f_a, e_a), (f_b, e_b) \gg := < e_a \mid f_b > + < e_b \mid f_a >, f_a, f_b \in \mathcal{F}, e_a, e_b \in \mathcal{F}^*$$

where $< \cdot \mid \cdot >$ denotes the duality product between $\mathcal{F}$ and $\mathcal{F}^*$.

Definition 2.3. *A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ is a Dirac structure if*

$$\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp} \tag{1}$$

where $^{\perp\!\!\!\perp}$ denotes the orthogonal complement with respect to $\ll \cdot, \cdot \gg$.

In the finite-dimensional case an equivalent, and often easier, characterization of Dirac structures is given as follows (see e.g. [7, 13] for a proof).

Proposition 2.4. *A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ is a Dirac structure if and only if the following two conditions are satisfied:*

$$\begin{aligned} &(i) < e \mid f >= 0, \quad \text{for all } (f, e) \in \mathcal{D} \\ &(ii) \dim \mathcal{D} = \dim \mathcal{F} \end{aligned} \tag{2}$$

[5] This definition can be extended [12, 8] to (non-constant) Dirac structures on manifolds: a Dirac structure $\mathcal{D}$ on a manifold $\mathcal{M}$ is defined as a vector subbundle of the Whitney sum $T\mathcal{M} \oplus T^*\mathcal{M}$ such that for each $m \in \mathcal{M}$ the linear space $D(m) \subset T_m\mathcal{M} \times T_m^*\mathcal{M}$ is a constant Dirac structure. This will be needed, for example, in the treatment of mechanisms in Section 3.3.

Note that the first equation in (2) can be regarded as a *power-conservation property*. Indeed, if f and e are power-conjugated variables, that is, $< e \mid f >$ denotes power, then the first equation of (2) states that the total power in the system is equal to zero.

A special type of Dirac structure is defined as follows:

Definition 2.5. *A Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ is separable if*

$$< e_a \mid f_b >= 0\,, \qquad \textit{for all } (f_a, e_a), (f_b, e_b) \in \mathcal{D} \tag{3}$$

Note that the definition of a Dirac structure already implies the weaker property

$$< e_a \mid f_b > + < e_b \mid f_a >= 0\,, \quad \text{for all } (f_a, e_a), (f_b, e_b) \in \mathcal{D} \tag{4}$$

Separable Dirac structures have the following simple geometric characterization.

Proposition 2.6. *Consider a separable Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$. Then*

$$\mathcal{D} = \mathcal{K} \times \mathcal{K}^\perp \tag{5}$$

for some subspace $\mathcal{K} \subset \mathcal{F}$, where $\mathcal{K}^\perp = \{e \in \mathcal{F}^* \mid < e \mid f >= 0, \forall f \in \mathcal{K}\}$. Conversely, any subspace $\mathcal{D}$ as in (5) for some subspace $\mathcal{K} \subset \mathcal{F}$ is a separable Dirac structure.

Proof. It is immediately seen that any subspace $\mathcal{K} \times \mathcal{K}^\perp$ satisfies (3), and is a Dirac structure since it satisfies a fortiori (4), and has dimension equal to $\dim \mathcal{F}$. Conversely, let the Dirac structure $\mathcal{D}$ satisfy (3). Define the following subspaces

$$\begin{aligned} \mathcal{F}_0 &= \{f \in \mathcal{F} \mid (f,0) \in \mathcal{D}\} & \mathcal{F}_1 &= \{f \in \mathcal{F} \mid \exists e \in \mathcal{F}^* \text{ s.t. } (f,e) \in \mathcal{D}\} \\ \mathcal{E}_0 &= \{e \in \mathcal{F}^* \mid (0,e) \in \mathcal{D}\} & \mathcal{E}_1 &= \{e \in \mathcal{F}^* \mid \exists f \in \mathcal{F} \text{ s.t. } (f,e) \in \mathcal{D}\} \end{aligned}$$

It is readily seen [9] that for any Dirac structure $\mathcal{E}_1 = (\mathcal{F}_0)^\perp, \mathcal{E}_0 = (\mathcal{F}_1)^\perp$. We will now show that (3) implies that $\mathcal{F}_0 = \mathcal{F}_1 =: \mathcal{K}$ (and hence $\mathcal{E}_0 = \mathcal{E}_1 =: \mathcal{K}^\perp$). Clearly, $\mathcal{F}_0 \subset \mathcal{F}_1$. Let now $(f_a, e_a) \in \mathcal{D}$ and thus $f_a \in \mathcal{F}_1$. Then for all $(f_b, e_b) \in \mathcal{D}$

$$\ll (f_a, 0), (f_b, e_b) \gg := < e_a \mid f_b > + < 0 \mid f_a >= < e_a \mid f_b >= 0$$

by (3). Hence, also $(f_a, 0) \in \mathcal{D}$ and thus $f_a \in \mathcal{F}_0$. By definition $\mathcal{F}_0 \times \mathcal{E}_0 \subset \mathcal{D}$, and hence $\mathcal{K} \times \mathcal{K}^\perp \subset \mathcal{D}$. Finally, since the dimension of $\mathcal{K} \times \mathcal{K}^\perp$ equals the dimension of $\mathcal{F}$ equality results. ∎

Note that (3) can be regarded as a generalized statement of Tellegen's theorem for electrical circuits (with f denoting the vector of currents, and e denoting the vector of voltages).

A typical form of a separable Dirac structure, which will be frequently used in the remainder, is the following.

Proposition 2.7. *Let $A : \mathcal{V} \to \mathcal{W}$ be a linear map between the linear spaces $\mathcal{V}$ and $\mathcal{W}$ with adjoint mapping $A^* : \mathcal{W}^* \to \mathcal{V}^*$, that is*

$$< w^* \mid Av >=< A^* w^* \mid v > \tag{6}$$

for all $v \in \mathcal{V}, w^* \in \mathcal{W}^*$ (where, as before, $< \cdot \mid \cdot >$ denotes the duality product between the dual spaces $\mathcal{W}$ and $\mathcal{W}^*$, respectively $\mathcal{V}$ and $\mathcal{V}^*$). Identify $(\mathcal{V} \times \mathcal{W})^* = \mathcal{V}^* \times \mathcal{W}^*$. Then

$$\begin{array}{rcl} \mathcal{D} & := & \{(v, w, v^*, w^*) \in (\mathcal{V} \times \mathcal{W}) \times (\mathcal{V}^* \times \mathcal{W}^*) \mid \\ & & Av = w, v^* = -A^* w^*\} \end{array} \tag{7}$$

is a separable Dirac structure.

Proof. Define $\mathcal{K} := \{(v, w) \in \mathcal{V} \times \mathcal{W} \mid Av = w\}$. Then $\mathcal{K}^\perp = \{(v^*, w^*) \in \mathcal{V}^* \times \mathcal{W}^* \mid v^* = -A^* w^*\}$. ∎

A key feature of Dirac structures is that their *composition* is again a Dirac structure (in contrast with symplectic or Poisson structures, where this is not generally the case). Let $\mathcal{D}_A \subset \mathcal{F}_A \times \mathcal{F}_c \times \mathcal{F}_A^* \times \mathcal{F}_c^*$ and $\mathcal{D}_B \subset \mathcal{F}_B \times \mathcal{F}_c \times \mathcal{F}_B^* \times \mathcal{F}_c^*$ be two Dirac structures with partially shared space of flow and effort variables $\mathcal{F}_c$, respectively $\mathcal{F}_c^*$. Define their *composition* as

$$\begin{array}{c} \mathcal{D}_A \circ \mathcal{D}_B = \{(f_A, e_A, f_B, e_B) \in \mathcal{F}_A \times \mathcal{F}_B \times \mathcal{F}_A^* \times \mathcal{F}_B^* \mid \exists (f, e) \in \mathcal{F}_c \times \mathcal{F}_c^* \text{ s.t.} \\ (f_A, e_A, f, e) \in \mathcal{D}_A, (f_B, e_B, -f, e) \in \mathcal{D}_A\} \end{array} \tag{8}$$

It has been shown in [7, 31] that the composition $\mathcal{D}_A \circ \mathcal{D}_B$ of any two Dirac structures is again a Dirac structure. Separable Dirac structures have the same compositional property:

Proposition 2.8. *Let $\mathcal{D}_A \subset \mathcal{F}_A \times \mathcal{F}_c \times \mathcal{F}_A^* \times \mathcal{F}_c^*$ and $\mathcal{D}_B \subset \mathcal{F}_B \times \mathcal{F}_c \times \mathcal{F}_B^* \times \mathcal{F}_c^*$ be two separable Dirac structures given as*

$$\mathcal{D}_i = \mathcal{K}_i \times \mathcal{K}_i^\perp, i = A, B,$$

where $\mathcal{K}_i \subset \mathcal{F}_i \times \mathcal{F}_c, i = A, B$. *Define the composition*

$$\mathcal{K}_A \circ \mathcal{K}_B = \{(f_A, f_B) \in \mathcal{F}_A \times \mathcal{F}_B \mid \exists f \in \mathcal{F}_c \text{ s.t. } (f_A, f) \in \mathcal{K}_A, (f_B, -f) \in \mathcal{K}_B\}$$

Then the composition $\mathcal{D}_A \circ \mathcal{D}_B$ *is the separable Dirac structure*

$$\mathcal{D}_A \circ \mathcal{D}_B = (\mathcal{K}_A \circ \mathcal{K}_B) \times (\mathcal{K}_A \circ \mathcal{K}_B)^{\perp} \tag{9}$$

For explicit equational representations of compositions of Dirac structures we refer to [7].

The compositionality property of Dirac structures is a key ingredient of port-Hamiltonian systems theory, which implies that the standard interconnection of port-Hamiltonian systems results in another port-Hamiltonian system with Dirac structure being the *composition* of the Dirac structures of the component port-Hamiltonian systems, and Hamiltonian equal to the *sum* of the Hamiltonians of the component systems.

2.4 The graph Dirac structures

We now have all ingredients to define Dirac structures corresponding to the incidence structure of a directed graph.

Definition 2.9. *Consider an open graph* $\mathcal{G}$ *with vertex, edge and boundary spaces, incidence operator* B *and boundary incidence operator* B_b. *The flow-continuous*[6] *graph Dirac structure* $\mathcal{D}_f(\mathcal{G})$ *is the following subspace of all flow and effort variables*

$$\begin{aligned} \mathcal{D}_f(\mathcal{G}) \;:=\; & \{(f_1, e^1, f_{0i}, e^{0i}, f_b, e^b) \in \\ & \Lambda_1 \times \Lambda^1 \times \Lambda_{0i} \times \Lambda^{0i} \times \Lambda_b \times \Lambda^b \mid \\ & B_i f_1 = f_{0i}, B_b f_1 = f_b, e^1 = -B_i^* e^{0i} - B_b^* e^b\} \end{aligned} \tag{10}$$

The effort-continuous graph Dirac structure $\mathcal{D}_e(\mathcal{G})$ *is*

$$\begin{aligned} \mathcal{D}_e(\mathcal{G}) \;:=\; & \{(f_1, e^1, f_0, e^0, f_b, e^b) \in \\ & \Lambda_1 \times \Lambda^1 \times \Lambda_0 \times \Lambda^0 \times \Lambda_b \times \Lambda^b \mid \\ & B_i f_1 = f_0, B_b f_1 = f_{0b} + f_b, e^1 = -B^* e^0, e^b = e^{0b}\} \end{aligned} \tag{11}$$

[6] The terminology *flow-continuous* and *effort-continuous* derives from the fact that in the first case the boundary flows f_b are exclusively linked to the edge flows f_1, while in the second case the boundary efforts e^b are determined by part of the internal vertex efforts e^0. Note that the space of involved flow and effort variables for $\mathcal{D}_f(\mathcal{G})$ and $\mathcal{D}_e(\mathcal{G})$ is *different*.

It directly follows from Proposition 2.7 that both $\mathcal{D}_f(\mathcal{G})$ and $\mathcal{D}_e(\mathcal{G})$ are separable Dirac structures. Note that $\mathcal{D}_f(\mathcal{G})$ and $\mathcal{D}_e(\mathcal{G})$ only differ in the role of the boundary flows and efforts, and that $\mathcal{D}_f(\mathcal{G}) = \mathcal{D}_e(\mathcal{G})$ if there are no boundary vertices.

2.5 Interconnection of open graphs and composition of graph Dirac structures

Interconnection of two open graphs $\mathcal{G}^\alpha$ and $\mathcal{G}^\beta$ is done by identifying some of their boundary vertices, and equating (up to a minus sign) the boundary efforts and flows corresponding to these boundary vertices, resulting in a new graph. For simplicity of exposition consider the case that the open graphs have *all* their boundary vertices in common, resulting in a (closed) graph with set of vertices $\mathcal{V}_i^\alpha \cup \mathcal{V}_i^\beta \cup \mathcal{V}$, where $\mathcal{V} := \mathcal{V}_b^\alpha = \mathcal{V}_b^\beta$ denotes the set of boundary vertices of both graphs.

The incidence operator of the interconnected (closed) graph is obtained as follows. For simplicity of notation consider the case that $\mathcal{R} = \mathbb{R}$. Let $\mathcal{G}^j$ have incidence operators

$$B^j = \begin{bmatrix} B_i^j \\ B_b^j \end{bmatrix}, \quad j = \alpha, \beta$$

The incidence operator B of the interconnected graph is then given as

$$B = \begin{bmatrix} B_i^\alpha & 0 \\ 0 & B_i^\beta \\ B_b^\alpha & B_b^\beta \end{bmatrix}, \tag{12}$$

corresponding to the interconnection constraints on the boundary potentials and currents given by

$$e^{b\alpha} = e^{b\beta}, \quad f_b^\alpha + f_b^\beta = 0 \tag{13}$$

Of course, several extensions are possible. For example, one may *retain* the set of shared boundary vertices $\mathcal{V}_b := \mathcal{V}_b^\alpha = \mathcal{V}_b^\beta$ as being boundary vertices (instead of internal vertices as above) by extending (13) to

$$e^{b\alpha} = e^{b\beta} = e^b, \quad f_b^\alpha + f_b^\beta + f_b = 0, \tag{14}$$

with f_b, e^b the boundary flows and efforts of the interconnected graph.

Comparing the *interconnection* of open graphs with the *composition* of their graph Dirac structures (see e.g. Proposition 2.8) it is readily seen that

the flow/effort-continuous graph Dirac structure of an interconnected graph equals the composition of the flow/effort-continuous graph Dirac structures of $\mathcal{G}^\alpha$ and $\mathcal{G}^\beta$; we leave the straightforward proof to the reader.

3 Port-Hamiltonian systems on graphs

In the first subsection we will describe how port-Hamiltonian dynamics can be defined with respect to the canonical graph Dirac structures defined above. In the subsequent subsections this will be illustrated on a number of typical examples, ranging from mass-spring systems and mechanisms to consensus algorithms.

3.1 Definition of port-Hamiltonian systems with regard to the graph Dirac structures

In this subsection we will apply the general definition of port-Hamiltonian systems with regard to an arbitrary Dirac structure, see e.g. [35, 32], to the canonical graph Dirac structures as defined above.

Consider for clarity of exposition throughout the effort-continuous graph Dirac structure $\mathcal{D}_e(\mathcal{G})$ involving the flow and effort variables

$$(f_1, e^1, f_0, e^0, f_b, e^b) \in \Lambda_1 \times \Lambda^1 \times \Lambda_0 \times \Lambda^0 \times \Lambda_b \times \Lambda^b$$

(The exposition is directly repeated for the flow-continuous graph Dirac structure $\mathcal{D}_f(\mathcal{G})$.) A port-Hamiltonian dynamics is specified by defining between all the *internal* conjugate flow and effort variables (f_1, e^1, f_0, e^0) either an *energy-storing* relation, or a purely *dissipative* relation. In general, an energy-storing relation between a vector of flow variables f and a conjugate vector of effort variables e is of the form

$$\dot{x} = -f, \quad e = \frac{\partial H}{\partial x}(x),$$

or dually

$$\dot{x} = e, \quad f = -\frac{\partial H}{\partial x}(x),$$

where x is a vector of energy variables (of the same dimension as f and e), and $H(x)$ is any function, representing the energy stored in the system.

Furthermore, a dissipative relation between a vector of flow variables f and a conjugate vector of effort variables e is any static relation

$$R(f, e) = 0$$

satisfying $< e \mid -f >\geq 0$ for all (f, e) satisfying $R(f, e) = 0$. Often the dissipative relation will be of an input-output type, that is, $f = -R(e)$ or dually $e = -R(f)$.

Thus a port-Hamiltonian dynamics is defined by adding to the linear relations imposed by the graph Dirac structure *constitutive* relations between all the internal effort and flow variables, either of energy-storing or of dissipative type[7] It is clear that this leaves many possibilities for defining port-Hamiltonian dynamics. In particular, energy-storage, respectively dissipation, can be associated to the vertices or to the edges, or to both. The examples presented in the next subsections cover a number of these different possibilities.

The interpretation of the flow/effort-continuous graph Dirac structure as describing *discrete conservation or balance laws* becomes more clear from the above description of port-Hamiltonian dynamics. For example, consider for the effort-continuous graph Dirac structure the case of energy storage associated to all the edges as well as to the all the vertices:

$$\begin{aligned} \dot{x}^1 &= e^1, \qquad & f_1 &= -\tfrac{\partial H^1}{\partial x^1}(x^1) \\ \dot{x}_0 &= -f_0, \qquad & e^0 &= \tfrac{\partial H_0}{\partial x_0}(x_0) \end{aligned}$$

for state variables $x^1 \in \Lambda^1$ and $x_0 \in \Lambda_0$, and energy functions H^1 and H_0. Then the relations imposed by the effort-continuous graph Dirac structure imply

$$\dot{x}_0 + B_i f_1 = 0, \quad \dot{x}^1 + B^* e^0 = 0$$

expressing discrete conservation (or balance) laws between the storage of the quantities x_0 associated to the vertices and the flow f_1 through the edges, respectively between the storage of the quantities x^1 associated to the edges and the effort e^0 at the vertices. The mass-spring system discussed in the next subsection will be of this type.

Finally, we note a fundamental property of any port-Hamiltonian dynamics. Let $H(x)$ denote the total energy of the port-Hamiltonian system. Then because of the power-conserving property of the Dirac structure, and denoting the flows and efforts of the dissipative elements by f_R, e^R,

$$\frac{d}{dt}H(x) =< \frac{\partial^T H}{\partial x}(x) \mid \dot{x} >=< e^R \mid f_R > + < e^b \mid f_b >\leq < e^b \mid f_b > \quad (15)$$

[7]Hence port-Hamiltonian dynamics generalizes both classical Hamiltonian dynamics (with no energy-dissipation), as well as gradient systems (where there is in general no oscillation between different energies and energy-dissipation does take place); see [34] and the references quoted therein.

Hence the total energy itself satisfies a conservation law: its increase is equal to the externally supplied power $< e^b \mid f_b >$ minus the dissipated power $- < e^R \mid f_R >$. Note that individual terms in the summation $< e^b \mid f_b >= \sum_{v_b} e^b(v_b) f_b(v_b)$ (summation over all boundary vertices v_b) have dimension of power but do not necessarily correspond to power; see the discussion in [42, 41].

3.2 Mass-spring systems

The first way of modeling a *mass-spring system* as a port-Hamiltonian system on a graph $\mathcal{G}$ is to associate the *masses* to the *vertices*, and the *springs* to the *edges* of the graph. (Alternatives will be discussed later on.)

Thus let us consider a graph $\mathcal{G}$ with N vertices (masses) and M edges (springs), specified by an incidence operator B. First consider the situation that the mass-spring system is located in one-dimensional space $\mathcal{R} = \mathbb{R}$, and the springs are scalar. A vector in the vertex space Λ_0 then corresponds to the vector p of the scalar momenta of all N masses, i.e., $p \in \Lambda_0 = \mathbb{R}^N$. Furthermore, a vector in the dual edge space Λ^1 will correspond to the total vector q of elongations of all M springs, i.e., $q \in \Lambda^1 = \mathbb{R}^M$. Next ingredient is the definition of the Hamiltonian (stored energy) $H : \Lambda^1 \times \Lambda_0 \to \mathbb{R}$, which splits for a mass-spring system into a sum of the kinetic and potential energies of each mass and spring

$$H(q,p) = \sum_v \frac{1}{2m_v} p_v^2 + \sum_e V_e(q_e)\,,$$

where p_v is the momentum corresponding to the mass associated to vertex v and q_e is the elongation corresponding to the spatial spring associated to edge e. The summations are over all vertices $v \in \mathcal{V}$ and all edges $e \in \mathcal{E}$. Furthermore, m_v are the mass parameters and V_e are the potential energies of the springs (e.g., in the case of a linear spring $V_e(q_e) = \frac{1}{2} k_e q_e^2$, with k_e the spring constant of the linear spring corresponding to edge e).

In the absence of boundary vertices the dynamics of the mass-spring system is then described as the port-Hamiltonian system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & B^T \\ -B & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} \tag{16}$$

defined with respect to the graph Dirac structure $\mathcal{D}_e(\mathcal{G}) = \mathcal{D}_f(\mathcal{G})$. Note that in fact the skew-symmetric matrix

$$J := \begin{bmatrix} 0 & B^T \\ -B & 0 \end{bmatrix} \tag{17}$$

defines a *Poisson structure* on the state space $\Lambda^1 \times \Lambda_0$.

The inclusion of boundary vertices, and thereby of external interaction, can be done in different ways. The first option is to associate *boundary masses* to the boundary vertices. Considering the effort-continuous graph Dirac structure $\mathcal{D}_e(\mathcal{G})$ we are then led to the port-Hamiltonian system

$$\begin{aligned} \dot{q} &= B^T \tfrac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -B \tfrac{\partial H}{\partial q}(q,p) + \begin{bmatrix} 0 \\ I \end{bmatrix} f_b \\ e^b &= \begin{bmatrix} 0 & I \end{bmatrix} \tfrac{\partial H}{\partial p}(q,p) \end{aligned} \tag{18}$$

where we have reordered the components of the momentum vector p in such a way that its last components correspond to the boundary masses. Here $f_b \in \Lambda_b$ are the external *forces* exerted (by the environment) on the boundary masses, and $e_b \in \Lambda^b$ are the *velocities* of these boundary masses.

Another possibility is to start from the flow-continuous graph Dirac structure $\mathcal{D}_f(\mathcal{G})$. In this case there are no masses associated to the boundary vertices, and we obtain the port-Hamiltonian system (with p_i the vector of momenta corresponding to the masses associated to the *internal* vertices, and Hamiltonian $H(q,p_i)$)

$$\begin{aligned} \dot{q} &= B_i^T \tfrac{\partial H}{\partial p_i}(q,p_i) + B_b^T e^b \\ \dot{p}_i &= -B_i \tfrac{\partial H}{\partial q}(q,p_i) \\ f_b &= B_b^T \tfrac{\partial H}{\partial q}(q,p_i) \end{aligned} \tag{19}$$

with $e^b \in \Lambda^b$ the velocities of the massless boundary vertices, and $f_b \in \Lambda_b$ the forces at the boundary vertices as *experienced* by the environment.

Note that in this latter case the external velocities e^b of the boundary vertices can be considered to be *inputs* to the system and the forces f_b to be *outputs*; in contrast to the previously considered case (boundary vertices corresponding to boundary masses), where the forces f_b are inputs and the velocities e^b the outputs of the system[8].

The above formulation of mass-spring systems in $\mathcal{R} = \mathbb{R}$ directly extends to $\mathcal{R} = \mathbb{R}^3$ by using the incidence operator $B = \hat{B} \otimes I_3$ and its adjoint as defined before.

[8] Of course, one could also consider the case where *some* of the boundary vertices are associated to masses while the remaining ones are massless. This will lead to a hybrid representation where the inputs consist of a part of the boundary forces and the complementary part of the boundary velocities (and the outputs are all the remaining variables).

Finally, we remark that in the above treatment we have considered springs with *arbitrary* elongation vectors $q \in \Lambda^1$. For ordinary, straight, springs the vector q of elongations is given as $q = B^T q_c$, where $q_c \in \Lambda^0$ denotes the vector of positions of the vertices. Hence in this case $q \in \operatorname{im} B^T \subset \Lambda^1$. Note that the subspace $\operatorname{im} B^T \times \Lambda_0 \subset \Lambda^1 \times \Lambda_0$ is an invariant subspace with regard to the dynamics (18) or (19). We will return to this issue in Subsection 4.1.

3.2.1 Mass-damper systems

The above set-up immediately extends to *mass-spring-damper systems* by associating *dampers* to some of the edges and springs to the remaining ones. For brevity we only consider the case of a pure *mass-damper system* (no springs), with massless boundary vertices. In this case the flow-continuous graph Dirac structure yields the following equations

$$\begin{aligned} B_i f_1 &= -\dot{p} \\ B_b f_1 &= f_b \\ e^1 &= -B_i^T \tfrac{\partial H}{\partial p}(p) - B_b^T e^b \end{aligned} \tag{20}$$

where f_1, e^1 are the flows and efforts corresponding to the dampers (damping forces, respectively, velocities). For example, for linear dampers the constitutive relations are given as $f_1 = -Re^1$, where R is a positive semi-definite diagonal matrix. Substitution into (20) then yields the port-Hamiltonian system[9]

$$\begin{aligned} \dot{p} &= -B_i R B_i^T \tfrac{\partial H}{\partial p}(p) - B_i R B_b^T e^b \\ f_b &= B_b R B_i^T \tfrac{\partial H}{\partial p}(p) + B_b^T R B_b^T e^b \end{aligned} \tag{21}$$

where, as before, the inputs e^b are the boundary velocities and f_b are the forces as experienced at the massless boundary vertices.

Note that the matrix

$$\mathcal{L} := \begin{bmatrix} B_i \\ B_b \end{bmatrix} R \begin{bmatrix} B_i^T & B_b^T \end{bmatrix}$$

[9] Recall again the generalized Hamiltonian nature of port-Hamiltonian systems. In this case the system has only one type of energy storage, i.e., kinetic energy, and thus no oscillatory behavior is present. In fact, the mass-damper system can be also represented as a *gradient dynamical system* with respect to an inner product defined by the mass parameters and a potential function defined by the damping parameters; see also [34] for further information on the relation between port-Hamiltonian and gradient systems.

is the *weighted Laplacian matrix* of the graph $\mathcal{G}$ (with weights given by the diagonal elements of R). It is well-known [5] that for a connected graph the matrix $\mathcal{L}$ has exactly one eigenvalue 0, with eigenvector $\mathbb{1}$, while all other eigenvalues are positive. This implies specific properties of the mass-damper dynamics, to which we will return in the context of the consensus algorithm (Subsection 3.4).

3.3 Three-dimensional mechanisms

This section presents an extension of mass-spring-damper systems in $\mathcal{R}$ or $\mathcal{R}^3$ to spatial mechanisms, that is, networks of rigid bodies related by joints. In this case, the linear space $\mathcal{R}$ is given by $\mathcal{R} := \mathrm{se}^*(3)$, the dual of the Lie algebra of the Lie group $SE(3)$ describing the position of a rigid body in $\mathbb{R}^3$.

A mechanism (or *multibody system*) is a mechanical system consisting of *rigid bodies* related by joints (defined as *kinematic pairs*) restricting the relative motion between the rigid bodies. The reader may find numerous references about their definition and analysis in [2] [29], using different geometric representations of rigid body displacements. In this paper however we shall follow closely the exposition in [22][15], which is based on the Lie group of isometries in Euclidean space $\mathbb{R}^3$.

The basic topology of the mechanism is described by a directed graph, called the *primary graph*, whose vertices correspond to the rigid bodies and whose edges are associated with the kinematic pairs. This is similar to the mass-spring or mass-damper systems described in Section 3.2, with the difference that the dynamical system associated with each vertex corresponds to rigid body dynamics instead of point-mass dynamics, and that the edges are in first instance associated with kinematic constraints between the bodies instead of springs or dampers. We shall see how (spatial) springs may be included in second instance.

3.3.1 The rigid body element

The configuration space of a rigid body is the Lie group of isometries in Euclidean space $\mathbb{R}^3$, called the Special Euclidean Group and denoted by $SE(3) \ni Q$ (also called the space of *rigid body displacements*). Using the momentum map associated with the action of $SE\,(3)$ on its cotangent bundle $T^*SE\,(3)$, following for instance [16, chap. 4], one may define the state space of the rigid body as $SE\,(3) \times se^*\,(3) \ni (Q, P)$ by means of the left trivialization, where P is called the *momentum in body frame*.

The kinetic energy of a rigid body is defined by

$$K(P) = \frac{1}{2}\langle P,\, \left(I^\flat\right)^{-1}(P)\rangle \tag{22}$$

where $I^\flat : se(3) \to se^*(3)$ is is a symmetric, positive-symmetric isomorphism, called the *inertia operator of the rigid body in the body frame.*The potential energy of the rigid body is defined by a function $U(Q)$ of the displacement Q. The potential energy may be due to gravity or may be zero in the case of the Euler-Poinsot problem.

We assume that the rigid body is subject to an external force which is expressed as an element $W_e \in se^*(3)$, called *force in fixed frame* [16] (also called *wrench in fixed frame* [15]) and obtained by the right trivialization of $T^*SE(3)$. We shall associate a conjugate velocity to this external force, the *velocity* of the body T_e *in fixed frame* [16] (also called the *twist in fixed frame* [15]), and obtained by the right trivialization of $TSE(3)$.

The dynamical equations of the rigid body elements may then be written as a port-Hamiltonian system [35] [19, eqn. (1.37)]:

$$\begin{aligned}
\tfrac{d}{dt}\begin{pmatrix} Q \\ P \end{pmatrix} &= \begin{pmatrix} 0 & TL_Q \\ -T^*L_Q & -P\times \end{pmatrix}\begin{pmatrix} dU(Q) \\ \left(I^\flat\right)^{-1}(P) \end{pmatrix} + \begin{pmatrix} 0 \\ Ad^*_Q \end{pmatrix} W_e \\
T_e &= \begin{pmatrix} 0 & Ad_Q \end{pmatrix}\begin{pmatrix} dU(Q) \\ \left(I^\flat\right)^{-1}(P) \end{pmatrix}
\end{aligned} \tag{23}$$

where TL_Q denotes the tangent map to the *left translation* (mapping the velocities $T \in se(3)$ in body frame into the velocities $v \in T_QSE(3)$), T^*L_Q denotes its dual map (mapping forces $F \in T^*_QSE(3)$ into forces in body frame $W \in se^*(3)$), Ad_Q denotes the *adjoint representation* (mapping velocities in body frame into velocities in fixed frame), Ad^*_Q denotes the adjoint map to Ad_Q, while finally $\times$ is defined by *the coadjoint representation of the Lie algebra* $se(3)$ as follows

$$W \times T = ad^*_T W$$

for any $(W,\, T) \in se^*(3) \times se(3)$. The Dirac structure of the port-Hamiltonian system (23) is thus specified as

$$\begin{aligned}
\mathcal{D}_{RB}(Q) \;=\; & \{(v,\, W,\, T_e,\, F,\, T,\, W_e) \in \\
& T_QSE(3) \times se^*(3) \times se(3) \times T^*_QSE(3) \times se(3) \times se^*(3) \,\text{s.t.} \\
& \begin{pmatrix} v \\ W \end{pmatrix} = \begin{pmatrix} 0 & TL_Q \\ -T^*L_Q & -P\times \end{pmatrix}\begin{pmatrix} F \\ T \end{pmatrix} + \begin{pmatrix} 0 \\ Ad^*_Q \end{pmatrix} W_e\,, \\
& T_e = \begin{pmatrix} 0 & Ad_Q \end{pmatrix}\begin{pmatrix} F \\ T \end{pmatrix}\Big\}
\end{aligned}$$

(24)

(Note that this is a non-constant Dirac structure [12, 8, 9] on $SE(3)$.)

In this way we have associated with every vertex of the primary graph of the mechanism the pair $(W_e, T_e) \in se^*(3) \times se(3)$ (i.e., the wrench and twist of the rigid body in fixed frame), and the dynamical system (23).

3.3.2 The kinematic pair

Constraints between the rigid bodies of the mechanism will be specified by *kinematic pairs* corresponding to each edge of the primary graph. A *kinematic pair* is the idealization of a set of contacts that occur between two rigid bodies at some configuration of the bodies. It constrains the possible relative twists between the bodies as well as the possible transmitted wrenches. The wrench W transmitted by a kinematic pair is constrained to a linear subspace of the space of wrenches $se^*(3)$ called the *space of constraint wrenches* and denoted by $\mathcal{CW}$. A relative twist between the two bodies is allowed by the kinematic pair when it produces no work with any transmissible wrench. The relative twist is thus constrained to a linear subspace $\mathcal{FT}$ of the space of twists $se(3)$, called the *space of freedom twists*. Since an ideal kinematic pair is workless the subspace $\mathcal{FT}$ is orthogonal (in the sense of the duality product) to the space of transmitted wrenches $\mathcal{CW}$, that is

$$\mathcal{FT} = \mathcal{CW}^{\perp} \tag{25}$$

We have defined the spaces of freedom twists and constraint wrenches as subspaces of of the Lie algebra $se(3)$ and its dual. However these spaces express constraints on the twists and wrenches of the rigid bodies related by the kinematic pairs and hence are expressed in some common frame with configuration Q_{KP}. (In most cases equal to the configuration of one of the related bodies.) Consequently, the constitutive relations of a kinematic pair is given in terms of its pair of twists and wrenches $(T_{KP}, W_{KP}) \in T_{Q_{KP}}SE(3) \times T^*_{Q_{KP}}SE(3)$ as follows

$$Ad^*_{Q_{KP}} W_{KP} \in \mathcal{CW} \quad \text{and} \quad Ad_{Q^{-1}_{KP}} T_{KP} \in \mathcal{FT} \tag{26}$$

Hence the constitutive equations of a kinematic pair may be expressed as the following non-constant separable Dirac structure:

$$\begin{aligned} \mathcal{D}_{\mathcal{CW}}(Q_{KP}) \;=\; & \Big\{ (T_{KP}, W_{KP}) \in T_{Q_{KP}}SE(3) \times T^*_{Q_{KP}}SE(3), \\ & Ad^*_{Q_{KP}} W_{KP} \in \mathcal{CW} \quad \text{and} \quad Ad_{Q^{-1}_{KP}} T_{KP} \in \mathcal{CW}^{\perp} \Big\} \end{aligned} \tag{27}$$

The kinematic pair introduced above represents ideal kinematic constraints; in general, however, mechanical work may be produced at the kinematic pair due to the presence of actuators or springs and dampers. Such an interaction is captured by considering the linear space $\mathcal{IW} := se^*(3)/\mathcal{CW}$ (which may be identified with a subspace of $se^*(3)$ complementary to the space of constraint wrenches $\mathcal{CW}$). The space of interaction twists is then defined as its dual space $\mathcal{IT} := \mathcal{IW}^* \simeq \mathcal{CW}^\perp$. Using the canonical projection π of $se^*(3)$ onto $\mathcal{IW}$, together with its adjoint map π^*, one may thus define an additional pair of port variables enabling to connect actuators, damper or spring elements to the kinematic pairs.

The resulting *interacting* kinematic pair is then defined as a 2-port element with constitutive relations defined by the following non-constant separable Dirac structure

$$\begin{aligned} \mathcal{D}^I_{\mathcal{CW}}(Q_{KP}) = \{(T_{KP},\, W_{KP},\, T_I,\, W_I) \in \\ T_{Q_{KP}}SE(3) \times T^*_{Q_{KP}}SE(3) \times \mathcal{CW}^\perp \times se^*(3)/\mathcal{CW}, \\ W_I = \pi \circ Ad^*_{Q_{KP}}(W_{KP}) \text{ and } T_{KP} = -Ad_{Q_{KP}} \circ \pi^*(T_I)\Big\} \end{aligned} \tag{28}$$

It is easy to check that if $W_I = 0$ then the interacting kinematic pair reduces to the kinematic pair as defined before.

3.3.3 The kinestatic connection network

The primary graph of the mechanism together with the kinematic pairs is called the *kinestatic model* of the mechanical system. Its associated Dirac structure is the *composition* of the Dirac structures corresponding to the kinematic pairs with the flow-continuous[10] graph Dirac structure of the primary graph.

Consider a mechanism defined by its primary graph $\mathcal{G}$ composed of n_{RB} *internal* vertices (associated with the rigid bodies), n_b boundary vertices corresponding to rigid bodies with zero inertia operator and n_{KP} edges (associated with the kinematic pairs). Define the vertex space $\Lambda_0 \ni T^{RB}$, and the edge space $\Lambda_1 \ni T^{KP}$ with respect to the Lie algebra $se(3)$, which represent respectively the external twist of the rigid bodies and the kinematic pairs. The dual spaces $\Lambda^0 \ni W^{RB}$, respectively $\Lambda^1 \ni W^{KP}$, then represent the external wrenches of the rigid bodies, respectively the wrenches of the

[10] Or the effort-continuous graph Dirac structure in case the rigid bodies corresponding to the boundary vertices have non-zero inertia operator.

kinematic pairs; see also Remark 2.1. The twists and wrenches of the boundary vertices (the rigid bodies with zero inertia operator) are associated with the vertex space $\Lambda_b \ni T^b$, respectively its dual $\Lambda^b \ni W^b$. Kirchhoff's laws on the twists and wrenches [10] amount to constraining these variables to belong to the flow-continuous graph Dirac structure, i.e.,

$$\left(T^{KP}, W^{KP}, T^{RB}, W^{RB}, T^b, W^b\right) \in \mathcal{D}_f(\mathcal{G})$$

Composition of $\mathcal{D}_f(\mathcal{G})$ with the Dirac structures $\mathcal{D}_{\mathcal{CW}}(Q_{KP})$ corresponding to all the kinematic pairs then results in the Dirac structure $\mathcal{D}_{KS}$ of the kinestatic model:

$$\left(T^I, W^I, T^{RB}, W^{RB}, T^b, W^b\right) \in \mathcal{D}_{KS} \tag{29}$$

3.3.4 The dynamics of a mechanism

The state space of the complete mechanism is the product space of the state spaces of all the rigid bodies, i.e.,

$$\mathcal{X} = (SE(3) \times se^*(3))^{n_{RB}} \ni x, \tag{30}$$

where n_B denotes the number of rigid bodies (equal to the number of internal vertices of the primary graph).

Recalling that the rigid body dynamics is defined as a port Hamiltonian system with respect to the Dirac structure (24) one then obtains the overall Dirac structure $\mathcal{D}_M$ of the mechanism by composing the Dirac structure $\mathcal{D}_{KS}$ of the kinestatic model with the Dirac structures $\mathcal{D}_{RB}$ of all the rigid bodies. Finally, defining the Hamiltonian $H_M(x)$ as the *sum* of the Hamiltonians of each body one obtains the following port-Hamiltonian model of the mechanism:

$$\left(-\frac{dx}{dt}, \frac{\partial H_M}{\partial x}(x), T^I, W^I, T^b, W^b\right) \in \mathcal{D}_M \tag{31}$$

3.4 Port-Hamiltonian formulation of consensus algorithms

While all previous examples of port-Hamiltonian dynamics on graphs arise from physical modeling the system treated in this subsection has a different origin. Nevertheless, it shares the same structure, and in fact, turns out to have dynamics equal to the mass-damper system treated before.

Consider a network of N agents moving in linear space $\mathcal{R}$, whose interaction topology is described by an *undirected* graph $\mathcal{G}$ (symmetric interaction).

Denote by $E(\mathcal{G})$ the edges of this undirected graph, consisting of unordered pairs (v, w) of vertices v, w. Hence $(v, w) \in E(\mathcal{G})$ if and only if $(w, v) \in E(\mathcal{G})$. Thus the vertices of the graph correspond to the agents, and the edges to the symmetric interactions between them. Distinguish between *leader* and *follower* agents, see e.g. [27], and associate the leader agents to the boundary vertices and the follower agents to the internal vertices.

Associated to each agent v there is a vector $x_v \in \mathcal{R}$ describing the motion in the linear space $\mathcal{R}$. In the standard consensus algorithm, see e.g. [23], the vector x_v of each follower agent v satisfies the following dynamics

$$\dot{x}_v(t) = - \sum_{(v,w)\in E(\mathcal{G})} g_{(v,w)}(x_v(t) - x_w(t)) \tag{32}$$

where $g_{(v,w)} > 0$ denotes a certain positive-definite *weight* matrix associated to each edge.

For simplicity of exposition let us take the linear space $\mathcal{R}$ to be equal to $\mathbb{R}$ in the rest of this section, implying that $g_{(v,w)} > 0$ are just positive constants. Collecting all *follower* variables x_v into one vector $x \in \mathbb{R}^{N_i}$, and all *leader* variables x_v into one vector $u \in \mathbb{R}^{N_b}$, it is readily checked that the dynamics can be written as

$$\dot{x} = -B_i G B_i^T x - B_i G B_b^T u \tag{33}$$

with B the incidence matrix of the graph *endowed with an arbitrary orientation*, and G the diagonal matrix with elements $g_{(v,w)}$ corresponding to each edge (v, w). This defines a port-Hamiltonian system with respect to the flow-continuous graph Dirac structure $\mathcal{D}_f(\mathcal{G})$. Indeed, consider the Hamiltonian function given by $H(x) := \frac{1}{2} \parallel x \parallel^2$ then (33) is equal to

$$\dot{x} = -B_i G B_i^T \frac{\partial H}{\partial x}(x) - B_i G B_b^T u \tag{34}$$

which are the same equations as for the mass-damper system (21), with $u = e_b \in \Lambda^b$, $y = f_b \in \Lambda_b$, and $\frac{\partial H}{\partial x}(x) = x$.

Using the properties of the Laplacian matrix, see the discussion at the end of Section 3.2.1, it can be shown [39, 33] that if the graph $\mathcal{G}$ is connected then for any $\bar{u}$ there exists a unique $\bar{x}$ satisfying $B_i G B_i^T \bar{x} + B_i G B_b^T \bar{u} = 0$, that is, an equilibrium of the system. Furthermore, this equilibrium is asymptotically stable.

Note that the corresponding artificial *output* vector

$$y := B_b G B_i^T \frac{\partial H}{\partial x}(x) + B_b G B_b^T u$$

equals minus the rate of the leader variables if the leader variables were supposed to obey the consensus algorithm with regard to the follower agents (which is *not* the case). Hence this artificial output measures the discrepancy between the leaders and the followers.

3.5 Port-Hamiltonian formulation of coordination control

Another example of port-Hamiltonian systems on a graph is provided by the coordination control of N dynamical systems [11]. Consider an open graph where each internal vertex corresponds to a port-Hamiltonian system. Coordination will be sought by designing a port-Hamiltonian dynamics associated to each edge of the graph.

For clarity of exposition we will restrict attention to the *flow-continuous* graph Dirac structure; the treatment for the effort-continuous graph Dirac structure runs analogously. Thus we consider an open graph $\mathcal{G}$, where to each internal vertex $v_i \in \mathcal{V}_i$ we associate a Dirac structure $\mathcal{D}_{v_i}$, a Hamiltonian $H_{v_i}(x_{v_i})$, and a resistive relation R_{v_i}. That is, for each internal vertex v_i we have a port-Hamiltonian system

$$(-\dot{x}_{v_i}, \frac{\partial H_{v_i}}{\partial x_{v_i}}, f^R_{v_i}, e^R_{v_i}, f_{v_i}, e_{v_i}) \in \mathcal{D}_{v_i} \tag{35}$$

with vectors of external flows and efforts f_{v_i}, e_{v_i}, where the flows and efforts $f^R_{v_i}, e^R_{v_i}$ corresponding to the resistive elements are related by the dissipative relation

$$R_{v_i}(f^R_{v_i}, e^R_{v_i}) = 0 \tag{36}$$

All these port-Hamiltonian systems are coupled to the flow-continuous graph Dirac structure $\mathcal{D}_f(\mathcal{G})$ by the standard interconnection

$$f_{v_i} = -f_{0i;v_i}, \quad e_{v_i} = e^{0i}_{v_i}, \quad v_i \in \mathcal{V}_i \tag{37}$$

where $f_{0i;v_i}$ and $e^{0i}_{v_i}$ are the components of the vector f_{0i}, respectively e^{0i}, corresponding to the internal vertex v_i. What results is a port-Hamiltonian system, with Dirac structure being the composition of the internal vertex Dirac structures $\mathcal{D}_{v_i}$ with the flow-continuous graph Dirac structure $\mathcal{D}_f(\mathcal{G})$, still having external flow and effort variables

$$f_1, e^1, f_b, e^b$$

[11] The set-up is heavily inspired by the innovative paper [1]. A major difference is that in our case the dynamics corresponding to the vertices and edges are all assumed to be port-Hamiltonian, instead of merely passive as in [1].

Coordination control is now sought by interconnecting the edge flow and effort variables f_1, e^1 to port-Hamiltonian systems attached to the *edges*. The simplest possibility for doing this is to associate energy storage to the edges, that is

$$f_1 = -\dot{x}_e, \quad e^1 = \frac{\partial H_e}{\partial x_e}(x_e), \quad e \in \mathcal{E}\,, \tag{38}$$

resulting in a port-Hamiltonian system with total Hamiltonian

$$H_{\text{tot}} = \sum_{v_i \in \mathcal{V}_i} H_{v_i}(x_{v_i}) + \sum_{e \in \mathcal{E}} H_e(x_e) \tag{39}$$

The aim is now to design H_e in such a manner that the minimum of H_{tot} corresponds to a desired set-point. Then under the additional assumption of enough dissipation in the system the system will converge to this desired configuration; see [1] for further details in a passivity framework.

4 Port-Hamiltonian systems on graphs obtained from symmetry reduction of symplectic Hamiltonian systems

In this section we will show how the port-Hamiltonian formulation of the mass-spring system on an open graph can be alternatively obtained from a canonical *symplectic* formulation by *symmetry reduction*, exploiting the invariance of the Hamiltonian function (in particular, of the spring potential energies).

4.1 Symmetry reduction from the symplectic formulation

Let us return to the formulation of a mass-spring system in Section 3.2, where the vertices correspond to the masses, and the edges to the springs in between them. An alternative is to consider the configuration[12] vector $q_c \in \Lambda^0 =: Q_c$, describing the *positions of all the masses.* In fact, this is the classical starting point for Lagrangian mechanics, where we do *not* start with the *energy variables* q and p, but instead we start with the configuration vector q_c and the corresponding velocity vector $\dot{q}_c$. The classical Hamiltonian formulation is then obtained by *defining* the vector of momenta $p \in \Lambda_0 = Q_c^*$

[12]Note that $q_c \in \Lambda^0$ is defined to be a function $q_c : \mathcal{V} \to \mathbb{R}$, assigning to each vertex its configuration in $\mathcal{R}$.

as $p = M\dot{q}_c$ (with M the diagonal mass matrix), resulting in the *symplectic phase space*

$$Q_c \times Q_c^* = \Lambda^0 \times \Lambda_0$$

For ordinary springs the relation between $q_c \in \Lambda^0$ and the vector $q \in \Lambda^1$ describing the elongations of the springs is given as $q = B^T q_c$. Hence in this case the Hamiltonian can be also expressed as a function H_c of (q_c, p) by defining

$$H_c(q_c, p) := H(B^T q_c, p) \tag{40}$$

It follows that the equations of motion of the mass-spring system (with boundary masses) are given by the canonical Hamiltonian equations

$$\begin{array}{rcl} \dot{q}_c & = & \frac{\partial H_c}{\partial p}(q_c, p) \\ \dot{p} & = & -\frac{\partial H_c}{\partial q_c}(q_c, p) + \begin{bmatrix} 0 \\ I \end{bmatrix} f_b \\ e^b & = & \begin{bmatrix} 0 & I \end{bmatrix} \frac{\partial H_c}{\partial p}(q_c, p) \end{array} \tag{41}$$

where, as before, f_b are the external forces exerted on the boundary masses and e_b are their velocities.

What is the relation with the port-Hamiltonian formulation given in Section 3.2 ? It turns out that this relation is precisely given by the standard procedure of *symmetry reduction* of a Hamiltonian system[13]. Indeed, since $B^T \mathbb{1} = 0$ the Hamiltonian function $H_c(q_c, p)$ given in (40) is *invariant* under the action of the group $\mathfrak{G} = \mathbb{R}$ acting on the phase space $\Lambda^0 \times \Lambda_0 = \mathbb{R}^{2N}$ by the symplectic group action

$$(q_c, p) \mapsto (q_c + \alpha \mathbb{1}, p)\,, \quad \alpha \in \mathfrak{G} = \mathbb{R} \tag{42}$$

From standard reduction theory, see e.g. [18] and the references quoted therein, it follows that we may factor out the configuration space $Q_c := \Lambda^0$ to the *reduced configuration space*

$$Q := \Lambda^0 / \mathfrak{G} \tag{43}$$

[13]This relation can be regarded as the discrete, graph-theoretic, version, of the correspondence between the port-Hamiltonian formulation of the Maxwell equations (using the Stokes-Dirac structure) and its symplectic formulation using the vector potential of the magnetic field, cf. [18, 40].

Let us first assume that the graph is *connected*, in which case we have the equality, cf. (54), $\ker B^T = \operatorname{span} \mathbb{1}$. Then we have the following identification

$$Q := \Lambda^0/\mathfrak{G} \simeq B^T\Lambda^0 \subset \Lambda^1 \tag{44}$$

Hence the *reduced state space* of the mass-spring system is given by $\operatorname{im} B^T \times \Lambda_0$, where $\operatorname{im} B^T \subset \Lambda^1$. (Note that $\operatorname{im} B^T = \Lambda^1$ if and only if the graph does not contain cycles.) Furthermore, under the symmetry action the canonical Hamiltonian equations (41) on the symplectic space $\Lambda^0 \times \Lambda_0$ reduce to the port-Hamiltonian equations (45) on $\operatorname{im} B^T \times \Lambda_0 \subset \Lambda^1 \times \Lambda_0$ obtained before:

$$\begin{aligned} \dot{q} &= B^T\dot{q}_c = B^T\tfrac{\partial H_c}{\partial p}(q_c,p) = B^T\tfrac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\tfrac{\partial H_c}{\partial q_c}(q_c,p) + \begin{bmatrix}0\\ I\end{bmatrix} f_b = -B\tfrac{\partial H}{\partial q}(q,p) + \begin{bmatrix}0\\ I\end{bmatrix} f_b \\ e^b &= \begin{bmatrix}0 & I\end{bmatrix}\tfrac{\partial H}{\partial p}(q,p) \end{aligned} \tag{45}$$

In case the graph is not connected, then the above symmetry reduction can be performed for each component of the graph (i.e., the symmetry group is $\mathbb{R}^{k_\mathcal{G}}$, with $k_\mathcal{G}$ denoting the number of components of the graph $\mathcal{G}$), yielding again the reduced state space $\operatorname{im} B^T \times \Lambda_0$.

Since the port-Hamiltonian equations (45) for a connected graph can thus be regarded to be the result of a 1-dimensional symmetry reduction, it follows from general reduction theory that a *conserved quantity* is present. Indeed the total momentum $p_{\text{tot}} := \mathbb{1}^T p$ satisfies the balance law

$$\frac{d}{dt}p_{\text{tot}} = \mathbb{1}_b^T f_b \tag{46}$$

In particular, p_{tot} is conserved if the sum of the external forces $\mathbb{1}_b^T f_b$ is equal to zero.

Remark 4.1. *In addition to the total momentum there may be other conserved quantities. Indeed, considering the Poisson structure matrix defined in (17) it follows that all quantities $\hat{q}$ with $B\hat{q} = 0$ (corresponding to the cycles of the graph $\mathcal{G}$) are conserved.*

5 The Kirchhoff-Dirac structure on graphs and its port-Hamiltonian dynamics

In this section we consider a third canonical Dirac structure on a graph, which corresponds to constraining the flows at the internal vertices to zero. Thus for the resulting port-Hamiltonian systems there is no energy-storage or dissipation associated with the vertices.

5.1 The Kirchhoff-Dirac structure

The *Kirchhoff-Dirac structure* is defined as

$$\begin{aligned}\mathcal{D}_K(\mathcal{G}) := \{(f_1, e^1, f_b, e^b) \in \Lambda_1 \times \Lambda^1 \times \Lambda_b \times \Lambda^b \mid \\ B_i f_1 = 0, B_b f_1 = f_b, \exists e^{0i} \in \Lambda^{0i} \text{ s.t. } e^1 = -B_i^* e^{0i} - B_b^* e^b\}\end{aligned} \tag{47}$$

Note that, in contrast to the flow/effort-continuous graph Dirac structures, the Kirchhoff-Dirac structure only involves the flow and effort variables of the *edge* and *boundary* vertex spaces (not of the internal vertex spaces).

Proposition 5.1. *$\mathcal{D}_K(\mathcal{G})$ is a separable Dirac structure.*

Proof. The Kirchhoff-Dirac structure is equal to the composition of the flow-continuous[14] graph Dirac structure $\mathcal{D}_f(\mathcal{G})$ with the following trivial separable Dirac structure defined on the internal vertex spaces

$$\{(f_{0i}, e^{0i}) \in \Lambda_{0i} \times \Lambda^{0i} \mid f_{0i} = 0\}$$

The result then follows from Proposition 2.8. ∎

Port-Hamiltonian dynamics with respect to the Kirchhoff-Dirac structure is defined completely similar as in the case of the flow/effort-continuous graph Dirac structure; the difference being that energy-storing or dissipative relations are now only defined for the flow and effort variables corresponding to the edges.

5.2 Electrical circuits

The example of a port-Hamiltonian system[15] with respect to a Kirchhoff-Dirac structure is an electrical RLC-circuit, with circuit graph $\mathcal{G}$. In this case the elements of Λ_1 and Λ^1 denote the vectors of currents through, respectively the voltages across, the edges. The Kirchhoff-Dirac structure in this case exactly amounts to the Kirchhoff's current and voltage laws (whence its name). Furthermore, the effort variables e^0 are the *potentials* at the vertices, while the boundary flows and efforts f_b, e^b are the *boundary currents*,

[14] Or the composition of the effort-continuous graph Dirac structure with $\{(f_0, e^0) \in \Lambda_0 \times \Lambda^0 \mid f_0 = 0\}$.

[15] Strictly speaking, the terminology 'port-Hamiltonian' is slightly confusing in this context, because 'ports' in electrical circuits are usually defined by *pairs of terminals*, that is *pairs* of boundary vertices with external variables being the currents through and the voltages across an edge corresponding to each such port. See also Remark 2.2.

respectively *boundary potentials* at the boundary vertices (the *terminals* of the electrical circuit).

On top of Kirchhoff's laws, the dynamics is defined by the energy-storage relations corresponding to either capacitors or inductors, and dissipative relations corresponding to resistors. The energy-storing relations for a capacitor at edge e are given by

$$\dot{Q}_e = -I_e, \quad V_e = \frac{dH_{Ce}}{dQ_e}(Q_e) \tag{48}$$

with Q_e the charge, and $H_{Ce}(Q_e)$ denoting the electric energy stored in the capacitor. Alternatively, in the case of an inductor one specifies the magnetic energy $H_{Le}(\Phi_e)$, where Φ_e is the magnetic flux linkage, together with the dynamic relations

$$\dot{\Phi}_e = V_e, \quad -I_e = \frac{dH_{Le}}{d\Phi_e}(\Phi_e) \tag{49}$$

Finally, a resistor at edge e corresponds to a static relation between the current I_e through and the voltage V_e across this edge, such that $V_e I_e \leq 0$. In particular, a linear (ohmic) resistor at edge e is specified by a relation $V_e = -R_e I_e$, with $R_e \geq 0$.

Alternatively, we can decompose the circuit graph $\mathcal{G}$ as the interconnection of a graph corresponding to the capacitors, a graph corresponding to the inductors, and a graph corresponding to the resistors. For simplicity let us restrict ourselves to the case of an LC-circuit without boundary vertices. Define $\hat{\mathcal{V}}$ as the set of all vertices that are adjacent to at least one capacitor *as well as* to at least one inductor. Then split the circuit graph into an open circuit graph $\mathcal{G}^C$ corresponding to the capacitors and an open circuit graph $\mathcal{G}^L$ corresponding to the inductors, both with set of boundary vertices $\hat{\mathcal{V}}$. Denote the incidence matrices of these two circuit graphs by

$$B^C := \begin{bmatrix} B_i^C \\ B_b^C \end{bmatrix}, \; B^L := \begin{bmatrix} B_i^L \\ B_b^L \end{bmatrix}$$

Assuming for simplicity that all capacitors and inductors are linear we arrive at the following equations for the C-circuit

$$\begin{aligned} B_b^C \dot{Q} &= I_b^C, \quad B_i^C \dot{Q} = 0, \\ B_b^{CT} \psi_b^C &= C^{-1} Q - B_i^{CT} \psi_i^C \end{aligned}$$

with Q the vector of charges of the capacitors and C the diagonal matrix with diagonal elements given by the capacitances of the capacitors. Similarly

for the L-circuit we obtain the equations

$$\begin{aligned}\dot{\Phi} &= B_b^{LT}\psi_b^L + B_i^{LT}\psi_i^L \\ 0 &= B_i^L L^{-1}\Phi \\ I_b^L &= -B_b^L L^{-1}\Phi\end{aligned}$$

with Φ the vector of fluxes and L the diagonal matrix of inductances of all the inductors.

The equations of the LC-circuit are obtained by imposing the interconnection constraints $\psi_b^C = \psi_b^L =: \psi_i$ and $I_b^C + I_b^L = 0$. By eliminating the boundary currents I_b^C, I_b^L one thus arrives at the differential-algebraic port-Hamiltonian equations

$$\begin{bmatrix} B_i^C & 0 \\ 0 & B_i^L \\ B_b^C & B_b^L \end{bmatrix} \begin{bmatrix} -\dot{Q} \\ L^{-1}\Phi \end{bmatrix} = 0$$

$$\begin{bmatrix} C^{-1}Q \\ -\dot{\Phi} \end{bmatrix} = \begin{bmatrix} B_i^{CT} & 0 & B_b^{CT} \\ 0 & B_i^{LT} & B_b^{LT} \end{bmatrix} \begin{bmatrix} \psi_i^C \\ \psi_i^L \\ \psi_i \end{bmatrix}$$

For a formulation of pure R, L or C circuits, and their weighted Laplacian matrices, we refer to [33].

5.3 Mass-spring systems with regard to a Lagrangian tree

An alternative port-Hamiltonian formulation of mass-spring(-damper) systems, in terms of the Kirchhoff-Dirac structure, can be given as follows. Recall the port-Hamiltonian formulation on $\Lambda^1 \times \Lambda_0$ with respect to the effort-continuous graph Dirac structure $\mathcal{D}_e(\mathcal{G})$, in which case the masses correspond to the vertices, and the springs to the edges of the graph $\mathcal{G}$, which we assume to be connected[16]. This graph can be extended to an *augmented* graph $\mathcal{G}_{\text{aug}}$ by adding a *ground vertex* g and adding edges from every vertex v of $\mathcal{G}$ towards this ground vertex. (The augmented graph is called a *Lagrangian tree*.) Furthermore, by *constraining* the effort e_g at the ground vertex to be zero we can equate the efforts e_v at every vertex v of $\mathcal{G}$ with the effort e_{vg} at the edge from v to g of the augmented graph $\mathcal{G}_{\text{aug}}$. In this way we can identify the effort-continuous graph Dirac structure $\mathcal{D}_e(\mathcal{G})$ with the Kirchhoff-Dirac structure $\mathcal{D}_K(\mathcal{G}_{\text{aug}})$ with the additional constraint

[16] For non-connected graphs $\mathcal{G}$ the same construction can be done for every connected component.

$e_g = 0$. (Note that this is again a separable Dirac structure since it equals the composition of the Kirchhoff-Dirac structure $\mathcal{D}_K(\mathcal{G}_{\text{aug}})$ with the trivial Dirac structure $\{(f_g, e_g) \mid e_g = 0\}$.)

In this way, the masses become associated with the edges e_{vg} from every vertex v to the ground vertex g. The interpretation of the ground vertex g is that it represents the reference point (with velocity e_g being zero). The flow f_g at the ground vertex g equals the total force exerted on a mass located at this reference point.

5.4 Hydraulic networks

The interpretation of the flow-/effort-continuous graph Dirac structures and the Kirchhoff-Dirac structure as capturing the basic conservation/balance laws of a network becomes especially tangible for hydraulic networks.

A hydraulic network can be modeled as a directed graph with edges corresponding to pipes, see e.g. [28, 11]. The vertices may either correspond to connection points with *fluid reservoirs* (buffers), or merely to connection points of the pipes. Let x_v be the stored fluid at vertex v and let q_e be the fluid flow through edge e. Collecting all stored fluids x_v into one vector x, and all fluid flows q_e into one vector q, the *mass-balance* is summarized as

$$\dot{x} = -Bq \tag{50}$$

with B denoting the incidence matrix of the graph. In the absence of fluid reservoirs this simply reduces to Kirchhoff's current laws $Bq = 0$.

For incompressible fluids a standard model of the fluid flow q_e through pipe e is

$$J_e \dot{q}_e = P_i - P_j - \lambda_e(q_e) \tag{51}$$

where P_i and P_j are the pressures at the tail, respectively head, vertices of edge e. Note that this encapsulates in fact *two* effects; one corresponding to energy storage and one corresponding to energy dissipation. Defining the energy variable $\varphi_e := J_e q_e$ the stored energy in the pipe associated with edge e is given as $\frac{1}{2J_e}\varphi_e^2 = \frac{1}{2}J_e q_e^2$. Secondly, $\lambda_e(q_e)$ is a damping force corresponding to energy dissipation.

In the case of fluid reservoirs at the vertices the pressures P_v at each vertex v are functions of x_v, and thus, being scalar functions, always derivable from an energy function

$$P_v = \frac{\partial H_v}{\partial x_v}(x_v), \quad v \in \mathcal{V}, \tag{52}$$

for some Hamiltonian $H_v(x_v)$ (e.g. gravitational energy). The resulting dynamics (with state variables x and q) is port-Hamiltonian with respect to the graph Dirac structure $\mathcal{D}_f(\mathcal{G}) = \mathcal{D}_e(\mathcal{G})$. The set-up is immediately extended to boundary vertices (either corresponding to controlled fluid reservoirs or direct in/ or out-flows).

In the absence of fluid reservoirs ($Bq = 0$) the dynamics reduces to a port-Hamiltonian system in the state variables q with respect to the Kirchhoff-Dirac structure .

Remark 5.2. *Similar models are used in inventory theory and supply-chain applications.*

5.5 Properties of the boundary flows and efforts of the Kirchhoff-Dirac structure

The fact that the internal vertex flows in the definition of the Kirchhoff-Dirac structure are all zero (and consequently no storage or dissipation at the vertices takes place) has a number of specific consequences for the behavior of the boundary flows and efforts (see [42] for closely related considerations).

Assume (for simplicity of exposition) that $\mathcal{R} = \mathbb{R}$. It is a well-known property [5] of any incidence matrix B that

$$\mathbb{1}^T B = 0 \tag{53}$$

where $\mathbb{1}$ denotes the vector with all components equal to 1. Furthermore [5], the rank of the incidence matrix is equal to the number of vertices minus the number of components [17] of the graph $\mathcal{G}$. In fact, each component of the graph satisfies the property

$$\ker B^T = \operatorname{span} \mathbb{1} \tag{54}$$

with B the (restricted) incidence matrix of this component and the dimension of $\mathbb{1}$ equal to its number of vertices. This implies several properties for the previously defined Dirac structures on the graph; we will concentrate on its consequences for the Kirchhoff-Dirac structure. From the definition of the Kirchhoff-Dirac structure it follows that

$$0 = \mathbb{1}^T B f_1 = \mathbb{1}_b^T B_b f_1 = -\mathbb{1}_b^T f_b \tag{55}$$

with $\mathbb{1}_b$ denoting the vector with all ones of dimension equal to the number of boundary vertices. Hence the boundary part of the Kirchhoff-Dirac structure

[17] A *component* is a maximal subgraph which is connected, that is, every two vertices are linked by a path of, -non-oriented-, edges.

of an open graph is constrained by the fact that the boundary flows add up to zero. Dually, we may always add to the vector of vertex efforts e^0 the vector $\mathbb{1}$ leaving invariant the edge efforts $e^1 = B^T e^0$. Hence, to the vector of boundary efforts e^b we may always add the vector $\mathbb{1}_b$.

Proposition 5.3. *Consider an open graph $\mathcal{G}$ with Kirchhoff-Dirac structure $\mathcal{D}_K(\mathcal{G})$. Then for each $(f_1, e^1, f_b, e^b) \in \mathcal{D}_K(\mathcal{G})$ it holds that*

$$\mathbb{1}_b^T f_b = 0\,,$$

while for any constant $c \in \mathbb{R}$

$$(f_1, e^1, f_b, e^b + c\mathbb{1}_b) \in \mathcal{D}_K(\mathcal{G})$$

This proposition implies that we may restrict the dimension of the space of boundary flows and efforts $\Lambda_b \times \Lambda^b$ of a connected graph by *two*. Indeed, we may define

$$\Lambda_{b\text{red}} := \{f_b \in \Lambda_b \mid f_b \in \ker \mathbb{1}_b^T\}$$

and its dual space

$$\Lambda^b_{\text{red}} := (\Lambda_{b\text{red}})^* = \Lambda^b / \operatorname{im} \mathbb{1}_b$$

It is straightforward to show that the Kirchhoff-Dirac structure $\mathcal{D}_K(\mathcal{G})$ reduces to a linear subspace of the reduced space $\Lambda_1 \times \Lambda^1 \times \Lambda_{b\text{red}} \times \Lambda^b_{\text{red}}$, which is also a Dirac structure. An interpretation of this reduction is that we may consider one of the boundary vertices, say the first one, to be the reference vertex, and that we may reduce the vector of boundary efforts $e^b = (e^{b1}, \cdots, e^{b\bar{b}})$ to a vector of voltages $(e^{b2} - e^{b1}, \cdots, e^{b\bar{b}} - e^{b1})$. A graph-theoretical interpretation is that instead of the incidence matrix B we consider the *restricted* incidence matrix [3].

For a graph $\mathcal{G}$ with more than one connected component the above holds for each connected component. It follows that there are as many independent constraints on the boundary flows f_b as the number of the connected components of the open graph $\mathcal{G}$. Dually, the space of allowed boundary efforts e^b is invariant under translation by as many independent vectors $\mathbb{1}_b$ as the number of connected components.

A complementary view on Proposition 5.3 is the fact that we may *close* an open graph $\mathcal{G}$ to a closed graph $\bar{\mathcal{G}}$ as follows. Consider first the case that $\mathcal{G}$ is connected. Then we may add one virtual ground vertex v_g, and virtual edges from this virtual vertex to every boundary vertex $v_b \in \mathcal{V}_e$, in such

a manner that the Kirchhoff-Dirac structure of this graph $\bar{\mathcal{G}}$ *extends* the Kirchhoff-Dirac structure of the open graph $\mathcal{G}$. In fact, to the virtual vertex v_g we may associate an arbitrary potential $e^0(v_g)$ (the ground-potential), and we may rewrite the externally supplied power $< e^b \mid f_b >$ as (since by (55) $\sum_{v_b} f_b(v_b) = 0$)

$$< e^b \mid f_b >= \sum_{v_b} (e^b(v_b) - e^0(v_g) f_b(v_b) = \sum_{v_b} e^{1b}(v_b) f_{1b}(v_b), \tag{56}$$

where $e^{1b}(v_b) := e^b(v_b) - e^0(v_g)$ and $f_{1b}(v_b) := f_b(v_b)$ denotes the effort across and the flow through the virtual edge towards the boundary vertex v_b. It is clear that for every element $(f_1, e^1, f_b, e^b) \in \mathcal{D}_K(\mathcal{G})$ corresponding to the open graph $\mathcal{G}$ there exists e^{1b} such that $(f_1, e^1, f_{1b}, e^{1b}) \in \bar{\mathcal{G}}$ for the closed graph $\bar{\mathcal{G}}$, and conversely for every $(f_1, e^1, f_{1b}, e^{1b}) \in \bar{\mathcal{G}}$ there exists e^b such that $(f_1, e^1, f_b, e^b) \in \mathcal{D}_K(\mathcal{G})$. This construction is extended to non-connected graphs by adding a ground vertex to *each* component containing boundary vertices.

6 Port-Hamiltonian dynamics on complexes

In this section the formulation of port-Hamiltonian dynamics on graphs will be extended to *k-complexes*. A main motivation for doing so is to be able to represent (continuous) *systems of conservation laws* directly on a discrete spatial domain by using *simplicial complexes*[18], in such a way as to retain the geometric nature of the involved variables.

This is closely related to an intense research activity on the use of mixed finite elements and discrete differential forms for the spatial discretization of partial differential equations, see among many other references e.g. [6, 14]. The novelty of the approach taken in this section resides in systematically taking into account the boundary interaction of such systems and in the resulting formulation of the discrete conservation laws in terms of Dirac structures and finite-dimensional port-Hamiltonian systems. More specifically, our approach can be regarded as the discrete analog of the formulation of systems of continuous conservation laws with open boundary as distributed-parameter (infinite-dimensional) port-Hamiltonian systems with respect to a special type of Dirac structure, the Stokes-Dirac structure, as performed in [36, 17]. In the absence of a boundary (or for fixed boundary conditions)

[18] Also other complexes, e.g. corresponding to cubes in $\mathbb{R}^3$ instead of tetrahedra, could be considered.

this is in its turn closely related to the extensive literature on the Hamiltonian formulation of systems of evolution equations using infinite-dimensional Poisson brackets; see e.g. [24] and the references quoted therein.

Apart from the main motivation of structure-preserving spatial discretization we will also discuss at the end of this section the extension of consensus algorithms to k-complexes.

6.1 From graphs to k-complexes

A directed graph with incidence matrix B can be regarded as an example of what is called a 1-*complex*. Indeed, the sequence

$$\Lambda_1 \stackrel{B}{\to} \Lambda_0 \stackrel{\mathbb{1}^T}{\to} \mathbb{R}$$

satisfies the property $\mathbb{1}^T \circ B = 0$. More generally, a k-complex Λ can be specified by a sequence of real linear spaces $\Lambda_0, \Lambda_1, \cdots, \Lambda_k$, together with a sequence of linear incidence operators

$$\Lambda_k \stackrel{\partial_k}{\to} \Lambda_{k-1} \stackrel{\partial_{k-1}}{\to} \cdots \Lambda_1 \stackrel{\partial_1}{\to} \Lambda_0$$

with the property that $\partial_{j-1} \circ \partial_j = 0$, $j = 2, \cdots, k$. The vector spaces Λ_j, $j = 0, 1 \cdots, k$, are called the spaces of j-*chains*. Each Λ_j is generated by a finite set of j-cells (like edges and vertices for graphs) in the sense that Λ_j is the set of functions from the j-cells to $\mathbb{R}$.

A typical example of a k-complex is the triangularization of a k-dimensional manifold, with the j-cells, $j = 0, 1, \cdots, k$, being the sets of vertices, edges, faces, etc. (the so-called simplicial complex).

Example 6.1. *Consider the triangularization of a* 2*-dimensional sphere by a tetrahedron with* 4 *faces,* 6 *edges, and* 4 *vertices. The matrix representation of the incidence operator* ∂_2 *(from the faces of the tetrahedron to its edges) is*

	$< v_1v_2v_3 >$	$< v_1v_3v_4 >$	$< v_1v_4v_2 >$	$< v_2v_4v_3 >$
$< v_1v_2 >$	1	0	-1	0
$< v_1v_3 >$	-1	1	0	0
$< v_1v_4 >$	0	-1	1	0
$< v_2v_3 >$	1	0	0	-1
$< v_2v_4 >$	0	0	-1	1
$< v_3v_4 >$	0	1	0	-1

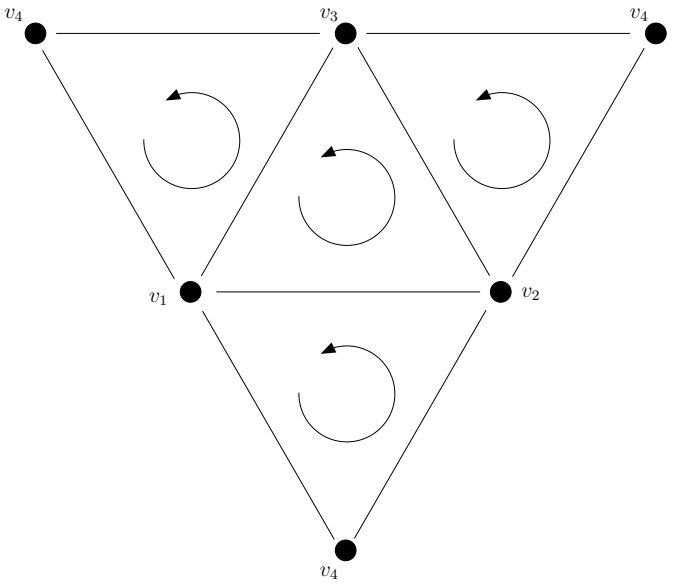

Figure 1: Tetrahedron triangularizing a sphere

where the expressions $< v_1v_2v_3 >, \ldots$ *denote the faces (with corresponding orientation), and* $< v_1v_2 >, \ldots$ *are the edges. The matrix representation of the incidence operator* ∂_1 *(from edges to vertices) is given as*

	$< v_1v_2 >$	$< v_1v_3 >$	$< v_1v_4 >$	$< v_2v_3 >$	$< v_2v_4 >$	$< v_3v_4 >$
$< v_1 >$	-1	-1	-1	0	0	0
$< v_2 >$	1	0	0	-1	-1	0
$< v_3 >$	0	1	0	1	0	-1
$< v_4 >$	0	0	1	0	1	1

It can be verified that $\partial_1 \circ \partial_2 = 0$.

Denoting the dual linear spaces by Λ^j, $j = 0, 1 \cdots, k$, we obtain the following dual sequence

$$\Lambda^0 \stackrel{d^1}{\to} \Lambda^1 \stackrel{d^2}{\to} \Lambda^2 \cdots \Lambda^{k-1} \stackrel{d^k}{\to} \Lambda^k$$

where the adjoint maps d^j $j = 0, 1 \cdots, k$, satisfy the analogous property $d^j \circ d^{j-1} = 0$, $j = 2, \cdots, k$. The elements of Λ^j are called *j-cochains.*

6.2 Canonical Dirac structures associated with open k-complexes and port-Hamiltonian systems

An *open* k-complex is identified by identifying a subset $\mathcal{V}^b_{(k-1)}$ of the set of all $(k-1)$-cells, called the *boundary* $(k-1)$-cells[19], while the remaining $(k-1)$-cells constitute the set $\mathcal{V}^i_{(k-1)}$ of *internal* $(k-1)$-cells. Define the linear space of functions from this subset of $(k-1)$-cells to $\mathbb{R}$ as $\Lambda_b \subset \Lambda_{k-1}$ with dual space denoted as Λ^b. Decompose correspondingly $\partial_k : \Lambda_k \to \Lambda_{k-1}$ as $\partial_k = (\partial_{k;i}, \partial_{k;b})$, with adjoint mapping $d^k = (d^{k;i}, d^{k;b})$. The space of flows and efforts $(f_b, e^b) \in \Lambda_b \times \Lambda^b$ describes the *distributed terminals* of the open k-complex.

Similar to the definition of the flow-continuous graph Dirac structure, see (10), we define the *flow-continuous k-complex Dirac structure* as the subspace

$$\begin{aligned}\mathcal{D}_f(\Lambda) := \{(f_k, e^k, f_{k-1;i}, e^{k-1;i}, f_b, e^b) \in \Lambda_k \times \Lambda^k \times \Lambda_{k-1;i} \times \Lambda^{k-1;i} \times \\ \Lambda_b \times \Lambda^b \mid \partial_{k;i} f_k = f_{k-1;i},\, \partial_{k;b} f_k = f_b,\, e^k = -d^{k;i} e^{k-1;i} - d^{k;b} e^b\}\end{aligned} \tag{57}$$

Similarly we can define the *effort-continuous k-complex Dirac structure* $\mathcal{D}_e(\Lambda)$ as the subspace

$$\begin{aligned}\mathcal{D}_e(\Lambda) := \{(f_k, e^k, f_{k-1;i}, e^{k-1;i}, f_b, e^b) \in \Lambda_k \times \Lambda^k \times \Lambda_{k-1} \times \Lambda^{k-1} \times \Lambda_b \times \Lambda^b \mid \\ \partial_k^i f_k = f_{k-1},\, \partial_{k;b} f_k = f_{k-1;b} + f_b,\, e^k = -d^{k;i} e^{k-1}, e^b = e^{k-1;b}\}\end{aligned} \tag{58}$$

As in the graph case, Proposition 2.7 implies that $\mathcal{D}_f(\Lambda), \mathcal{D}_e(\Lambda)$ are separable Dirac structures. Furthermore, the linear subspace

$$\begin{aligned}\mathcal{D}_K(\Lambda) := \{(f_k, e^k, f_b, e^b) \in \Lambda_k \times \Lambda^k \times \Lambda_b \times \Lambda^b \mid \\ \partial_{k;i} f_k = 0,\, \partial_{k;b} f_k = f_b,\, \exists e^{k-1;i} \text{ s.t. } e^k = -d^{k;i} e^{k-1;i} + d^{k;b} e^b\}\end{aligned} \tag{59}$$

defines a separable Dirac structure, called the *Kirchhoff-Dirac structure on k-complexes*; the proof is similar to Proposition 5.1. Analogously to the graph case, the Kirchhoff-Dirac structure for open k-complexes implies certain constraints on the incoming 'currents' f_b. Indeed, by the fact that $\partial_{k-1} \circ \partial_k = 0$ it follows that $\partial_{(k-1);b} f_b = 0$, where $\partial_{(k-1);b}$ denotes the $(k-1)$-th incidence operator restricted to $\Lambda_b \subset \Lambda_{k-1}$. (Note that in the

[19] One could also consider as boundary cells subsets of the j-th cells for $j \neq k-1$. In particular, choosing $j = k$ would correspond to 'distributed interaction'. The choice $j = k-1$ corresponds to the important case of 'boundary' interaction.

case of a graph the role of $\partial_{(k-1);b}$ is played by the linear map $\mathbb{1}_b^T$.) As in the case of graphs, this allows us to *reduce* the Kirchhoff behavior to a space that is still a Dirac structure, or, alternatively, to *close* the open k-complex. This is done by completing the open k-complex Λ with space of boundary currents Λ_b by an additional set of $(k-1)$-cells and k-cells.

Similarly to open graphs , we can define *interconnections* of open complexes. The most common form of interconnection of two open k-complexes is through shared boundary $(k-1)$-cells, similarly to the interconnection of open graphs through shared boundary vertices [20]. 'Shared' here means that the $(k-1)$-subcomplexes of both open k-complexes defined by the boundary $(k-1)$-cells are the same.

Similarly to the graph case, port-Hamiltonian dynamics on a k-complex can be defined in various ways; either with respect to its flow/effort-continuous k-complex Dirac structure, or with respect to its Kirchhoff-Dirac structure. The next sections will describe a couple of examples.

6.3 2D transverse magnetic Maxwell's equations defined on a 2-complex

Consider the transverse magnetic Maxwell equations with respect to some plane in physical space. Such a situation arises when modeling a thin isotropic dielectric film comprised between two planar conductors, while neglecting boundary effects at the boundaries of the conducting plates. The electrical field may then be approximated as being orthogonal to the plane and the magnetic field as being tangential to the plane. Various discretization methods have been used for these transverse Maxwell's equations. Here instead we shall directly obtain a discretized model expressed as a port-Hamiltonian system defined with respect to the flow-continuous 2-complex Dirac structure.

First we start with the simplicial 2-complex defined by a triangularization of the plane. Using the planar symmetry of the field, we shall associate different chains and cochains with the electrical and magnetic fields.

Because of the assumption of orthogonality of the electrial field with the plane, we identify the *electric field induction* with a 2-*cochain* $D \in \Lambda^2$ and the *electric field intensity* with a 2-*chain* $E \in \Lambda_2$. Furthermore, using the assumption that the magnetic field is tangential to the plane, we identify

[20] In principle there are several possibilities for the interconnection of k-complexes, depending also on the definition of the boundary cells. For example, one may also define the interconnection of a k-complex with an l-complex through a number of shared boundary p-cells, with $p \leq k, p \leq l$.

the *magnetic field induction* with a 1-*chain* $B \in \Lambda_1$ and the *magnetic field intensity* with a 1-*cochain* $H \in \Lambda^1$.

Maxwell's equations applied to the discrete fields defined above may then be expressed, in the absence of a boundary, as the following two relations (conservation/balance laws)

$$\begin{aligned} \frac{dD}{dt} &= d^2 H \\ \frac{dB}{dt} &= -\partial_2 E \end{aligned} \tag{60}$$

The properties of the dielectric is defined by the positive-definite electric permittivity tensor ϵ on the space of electrical intensity 2-chains and by the positive-definite magnetic permeability tensor μ on the space of the magnetic field intensity 1-cochains. With an abuse of notation the map induced by the permittivity tensor from the electric intensity 2-chains to the electric induction 2-cochains will be also denoted by ϵ. In the same way we use μ also for the induced map from the magnetic intensity 1-cochains to the magnetic induction 1-chains. This leads to the following constitutive relations

$$D = \epsilon\, E \quad B = \mu\, H \tag{61}$$

and inversely

$$E = \epsilon^{-1} D \quad H = \mu^{-1} B \tag{62}$$

Note that the discrete electric permittivity and magnetic permeability tensors depend not only on the continuous properties of the material but also on the geometry of the meshing (triangularization). Furthermore, these tensors define the electromagnetic energy as (we use $\mathbb{H}$ for the Hamiltonian function in order to distinguish it from the magnetic field intensity H)

$$\mathbb{H}(D, B) = \frac{1}{2} \left\langle \epsilon^{-1} D,\, D \right\rangle + \frac{1}{2} \left\langle \mu^{-1} B,\, B \right\rangle \tag{63}$$

where $\langle\, ,\, \rangle$ denotes the duality pairing between dual vector spaces. Note that $E = \frac{\partial \mathbb{H}}{\partial D}(D, B)$ and $H = \frac{\partial \mathbb{H}}{\partial B}(D, B)$.

Introducing boundary edges (1-cells) we are then led to the following discrete Maxwell's equations

$$\begin{aligned} \frac{dD}{dt} &= d^{2;i} \frac{\partial \mathbb{H}}{\partial B}(D, B) + d^{2;b} e^b \\ \frac{dB}{dt} &= -\partial_{2;i} \frac{\partial \mathbb{H}}{\partial D}(D, B) \\ f_b &= \partial_{2;b} \frac{\partial \mathbb{H}}{\partial D}(D, B) \end{aligned} \tag{64}$$

which defines a port-Hamiltonian system with respect to the flow-continuous 2-complex Dirac structure[21], and with boundary port variables f_b and e^b being respectively the magnetic flux and magnetic field intensity at the boundary edges of the system.

6.4 Diffusive systems on k-complexes

In this subsection we will consider a particular type of port-Hamiltonian dynamics based on the flow-continuous k-complex Dirac structure, by specifying dissipative relations between $\Lambda_{k-1;i}$ and $\Lambda^{k-1;i}$, together with one type of energy-storage relations between Λ_k and Λ^k. The resulting class of systems will be called *diffusive systems*.

Consider a k-complex, together with its flow-continuous Dirac structure $\mathcal{D}_\Lambda$. We associate to every k-cell an energy storage, by imposing the following dynamical energy-storing relation between f_k and e^k

$$-\dot{x} = e^k, \quad f_k = \frac{\partial H}{\partial x}(x), \quad x \in \Lambda^k \tag{65}$$

with $H(x)$ the total stored energy, and $x \in \Lambda^k$ the total vector of energy variables. Furthermore, to every internal $(k-1)$-cell we associate a (linear) resistive relation, leading to

$$e^{k-1;i} = -R f_{k-1;i}, \quad R = R^T \geq 0 \tag{66}$$

Then we arrive at the dynamics

$$\begin{array}{rcl} \dot{x} & = & -d^{k;i}\, R\, \partial_{k;i} \frac{\partial H}{\partial x}(x) - d^{k;b} e^b \\ f_b & = & -\partial_{k;b} \frac{\partial H}{\partial x}(x) \end{array} \tag{67}$$

This defines a port-Hamiltonian system with inputs e^b and outputs f_b, which can be regarded as a standard model for a discretized *diffusive system*. For the continuous analog of this diffusive system, using differential forms instead of (co-)chains, we refer to [30].

As a typical example consider the discrete formulation of a mass-diffusion process on a bounded 3-dimensional spatial domain Z with boundary ∂Z (e.g., the diffusion of a chemical species in a medium due to the gradient of its chemical potential [4, chap. 17]); see [37] for the example of discretized heat

[21] Note that, in order to stick to the same sign convention as in the continuous Maxwell's equations, the sign convention for the flow-continuous 2-complex Dirac structure is just the opposite to the one used above; see (57).

flow. Discretize the spatial domain Z by triangularization, corresponding to a standard simplicial complex consisting of the sequence of cells :

$$\underset{\text{tetrahedra}}{\Lambda_3} \overset{\partial_3}{\to} \underset{\text{faces}}{\Lambda_2} \overset{\partial_2}{\to} \underset{\text{edges}}{\Lambda_1} \overset{\partial_1}{\to} \underset{\text{vertices}}{\Lambda_0} \overset{\partial_0}{\to} \mathbb{R} \tag{68}$$

To every tetrahedron we may now associate a variable denoting the total mass (of the chemical species) in this tetrahedron, leading to a vector $x \in \Lambda^3$. Bookkeeping of the ingoing and outgoing mass flow in every tetrahedron corresponds to the *mass balance equation* (a typical example of a discrete conservation law)

$$\dot{x} = d^{3;i} N^i + d^{3;b} N^b,$$

where $N^i \in \Lambda^{2i}$ denotes the mass flux of the chemical species through the internal faces and $N^b \in \Lambda^{2b}$ is the mass flux through the boundary faces (corresponding to a triangularization of the 2-dimensional boundary ∂Z). Associated to x there is the Hamiltonian $H(x)$ (the free Gibbs energy), defining the vector $\mu = \frac{\partial H}{\partial x}(x) \in \Lambda_3$ of chemical potentials, which is conjugate to the vector $\dot{x} \in \Lambda^3$. Furthermore, the expressions

$$F_i = \partial_{3;i} \mu \in \Lambda_{2;i}, \quad F_b = \partial_{3:b} \mu \in \Lambda_{2;b}$$

define the *generating functions* (resulting from non-equilibrium) at the internal, respectively boundary, faces. Finally the internal generating functions determine the internal mass flux N^i, by a dissipative relation of the form

$$N^i = -RF_i$$

for a certain matrix $R = R^T \geq 0$. All this leads to the finite-dimensional port-Hamiltonian system

$$\begin{array}{rcl} \dot{x} & = & -d^{3;i} R \partial_{3;i} \frac{\partial H}{\partial x}(x) + d^{3;b} N^b \\ F_b & = & \partial_{3;b} \frac{\partial H}{\partial x}(x) \end{array} \tag{69}$$

modeling a diffusive system with input N^b (mass flux through the boundary faces), and output F_b (generating function at the boundary faces).

6.5 Generalized consensus algorithms on k-complexes

Note that the above discretized diffusion model is in some sense opposite to the formulation of the standard consensus algorithm on a graph (or,

equivalently, the model of a mass-damper system) treated before. Indeed, in the latter case, energy-storage is associated with the vertices (0-cells), while dissipation is associated with the edges (1-cells). This indicates that there are different possibilities to extend consensus algorithms from graphs to a higher-dimensional situation. Indeed, we may as well consider the generalized consensus algorithm corresponding to a diffusive system without boundary

$$\dot{x} = -d_k R \partial_k x, \quad x \in \Lambda^k \simeq \Lambda_k \tag{70}$$

with R a positive diagonal matrix of appropriate dimensions, or, alternatively,

$$\dot{z} = -\partial_k G d_k z, \quad z \in \Lambda_{k-1} \simeq \Lambda^{k-1} \tag{71}$$

with G a positive diagonal matrix. The dynamical properties of both possibilities are rather different. For example, in the case $k = 2$ the state $x \in \Lambda^2 \simeq \Lambda_2$ for the first algorithm will converge to the subspace $\ker \partial_2$, while for the second algorithm $z \in \Lambda_1 \simeq \Lambda^1$ will converge to the subspace $\ker d_2$ which contains the whole subspace $\operatorname{im} d_1$ (and will be equal to it if the Betti numbers are zero).

As an example, consider the triangularization of a 2-sphere by a single tetrahedron, with incidence structure given before. Compute $\partial_2 G d_2$ as

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \end{bmatrix} =$$
$$\begin{bmatrix} g_1+g_3 & -g_1 & -g_3 & g_1 & g_3 & 0 \\ -g_1 & g_1+g_2 & -g_2 & -g_1 & 0 & g_2 \\ -g_3 & -g_2 & g_2+g_3 & 0 & -g_3 & -g_2 \\ g_1 & -g_1 & 0 & g_1+g_4 & -g_4 & g_4 \\ g_3 & 0 & -g_3 & -g_4 & g_3+g_4 & -g_4 \\ 0 & g_2 & -g_2 & g_4 & -g_4 & g_2+g_4 \end{bmatrix} \tag{72}$$

the generalized consensus algorithm $\dot{z} = -\partial_2 G d_2 z$ converges to the 3-dimensional subspace $\ker d_2 = \operatorname{im} d_1$.

On the other hand, the generalized consensus algorithm corresponding to the diffusive system $\dot{x} = -d_2 R \partial_2 x$ is given as[22]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = - \begin{bmatrix} r_1 + r_2 + r_4 & -r_2 & -r_1 & -r_4 \\ -r_2 & r_2 + r_3 + r_6 & -r_3 & -r_6 \\ -r_1 & -r_3 & r_1 + r_3 + r_5 & -r_5 \\ -r_4 & -r_6 & -r_5 & r_4 + r_5 + r_6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \tag{73}$$

which converges to the 1-dimensional subspace span $\mathbb{1}$ in $\mathbb{R}^4$.

7 Conclusions

We have laid down a general geometric framework for the description of physical networks dynamics on graphs. Starting point are the conservation laws corresponding to the incidence matrix of the graph. This defines three canonical Dirac structures on the combined vertex, edge, and boundary spaces and their duals, where the last one (the Kirchhoff-Dirac structure) corresponds to the absence of energy storage or energy dissipation at the vertices. Relating the internal flows and efforts by either energy-storing or dissipative relations yields various forms of port-Hamiltonian dynamics. We have illustrated the approach on a number of typical physical examples. Other examples that have not been discussed include e.g. supply-chain models and compartmental systems. Furthermore we have shown how examples from a different origin, such as consensus algorithms and coordination control strategies, can be formulated and analyzed within the same framework. In a future paper we will explore the potential of the framework for the modeling and analysis of chemical reaction networks, which cannot directly be modeled by standard directed graphs.

In the last part of the paper we have extended the framework from directed graphs to general k-complexes. We have shown how this allows to a give a spatially discretized model of the 2-D Maxwell equations and of general diffusive systems. Pertinent questions concern the relation of these

[22] This dynamics turns out to be equal to the *standard* consensus algorithm on the complete graph with 4 vertices. This can be also seen by associating to each of the faces of the 2-complex a vertex, and by assigning an edge between two vertices if the corresponding faces are adjacent. Since in the case of a single tetrahedron all faces are adjacent, a complete graph results. This yields a correspondence between the operator ∂_2 and the transposed incidence matrix B^T of the complete graph.

methods to structure-preserving spatial discretization methods for their description by partial differential equations models. Furthermore, we have indicated how the discrete diffusive systems models may motivate alternative consensus algorithms.

For clarity of exposition we have only considered the basic building blocks of port-Hamiltonian systems and graphs. Indeed, because the *interconnection* of port-Hamiltonian systems again defines a port-Hamiltonian system, the framework also covers heterogeneous and multi-scale situations, where several of the constructs considered in the present paper are connected to each other.

The models treated in this paper all correspond to conservation/balance laws within a particular physical domain. Furthermore, the energy-balance of the system components can be seen to *result* from the underlying conservation laws and the assumption of integrable constitutive relations for energy-storage. On the other hand, port-based (bond-graph) modeling as originating in the work of Paynter [26] is aimed at providing a unifying modeling framework for multi-physics systems, by directly *starting* from energy-flows between system components from different physical domains. This also results in port-Hamiltonian models as has been amply demonstrated in e.g. [20, 21, 35, 32, 13]. It is well-known that bond-graph modeling involves an additional abstraction step (e.g., different electrical circuits may lead to the same bond-graph, and, conversely, different bond-graphs may correspond to the same electrical circuit). Furthermore, in the case of electrical circuits port-based modeling starts with a *port* description (pairs of terminals), instead of the more basic starting point of *terminals* corresponding to conservation laws. Although in most situations the resulting port-Hamiltonian systems are the same this leaves some questions to be answered; see also [42].

Finally, a main motivation for physical system network modeling, apart from its importance in analysis and simulation, is its use for *control* and *design* purposes. Port-Hamiltonian systems theory has been successful in exploiting the physical structure for control and design purposes, see e.g. [32, 25], using various forms of passivity-based control, control by interconnection, and tools originating in network synthesis theory. The applications of this control methodology to the broad range of port-Hamiltonian systems on graphs and k-complexes as formulated in this paper is an interesting area for further research.

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